

LECTURER NOTE-2020
BRANCH-ELECTRICAL ENGINEERING
SUBJECT-CONTROL SYSTEM ENGG.
SEMESTER-6th

CHAPTER-4, FREQUENCY RESPONSE ANALYSIS

What is Frequency Response?

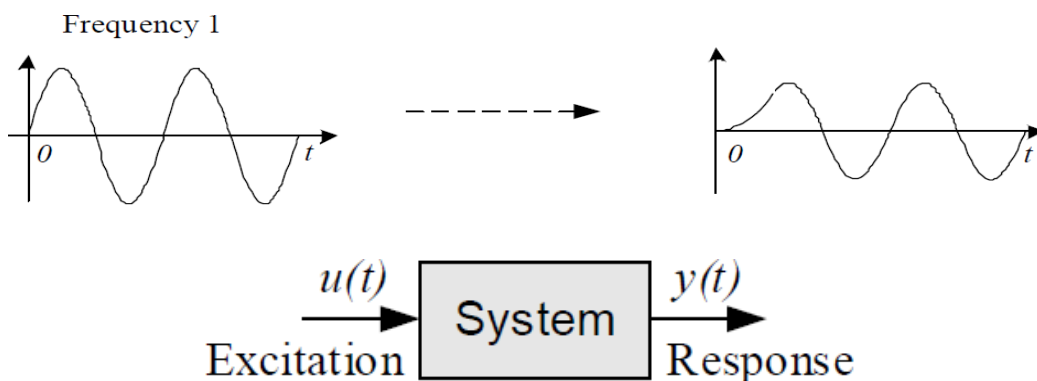
Consider a system with a sinusoidal input

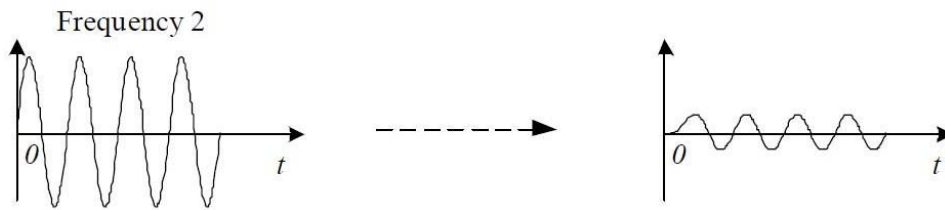
$$r(t) = A \sin \omega t$$

Under steady state, the system output as well as signals at all other points in the system are sinusoidal. The steady state output may be written as

$$c(t) = B \sin (\omega t + \phi)$$

The magnitude and phase relationship between the sinusoidal input and the steady state output of a system is termed as frequency response. In linear time-invariant systems, the frequency response is independent of the amplitude and phase of the input signal.





Advantages of Frequency Response Analysis:

The frequency response test on a system or a component is normally performed by keeping the amplitude A fixed and determining B and ϕ for a suitable range of frequencies. Signal generators and precise measuring instruments are readily available for various ranges of frequencies and amplitudes.

- Whenever it is not possible to obtain the form of the transfer function of a system through analytical techniques, the necessary information to compute the transfer function can be extracted by performing the frequency response test on the system.
- The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out somewhat more easily in frequency domain than in time domain.
- The effect of noise disturbance and parameter variations are relatively easy to visualize and assess through frequency response.
- The Nyquist Stability criterion is a powerful frequency domain method of extracting the information regarding stability as well as relative stability of a system without the need to find roots of the characteristic equation.

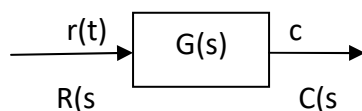
How to obtain Steady-State Outputs to Sinusoidal Inputs?

The Laplace Transform of the output of a linear single-input, single-output system with transfer function $G(s)$ can be expressed in terms of the input as

$$C(s) = G(s) \cdot R(s)$$

We know, in general that $s = \sigma + j\omega$. However, it will be shown here that, for sinusoidal steady-state analysis, we shall replace s by its imaginary component $j\omega$ only, since in steady state, the contribution of the real part will disappear for a stable system.

Consider the stable, linear system shown below.



Let us assume that the input signal $r(t) = A \sin \omega t$.

Suppose that the transfer function $G(s)$ of the system can be written as a ratio of two polynomials in s as

$$G(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s+a)(s+b)(s+c)}$$

The Laplace Transform of the output of the system is then $G(s)R(s) = \frac{P(s)}{q(s)}R(s)$, where $R(s)$ is the Laplace Transform of the input $r(t)$.

$$C(s) = G(s)R(s) = G(s) \frac{A\omega}{s^2 + \omega^2}$$

$$= \frac{K_1}{s+a} + \frac{K_2}{s+b} + \frac{K_3}{s+c} + \frac{K}{s+j\omega} + \frac{K^*}{s-j\omega}$$

Where, K^* is the conjugate of K . The inverse Laplace of the above equation yields

$$c(t) = K_1 e^{-at} + K_2 e^{-bt} + K_3 e^{-ct} + K e^{-j\omega t} + K^* e^{j\omega t}$$

For a stable system, a, b, c have positive real parts. Hence, as t approaches ∞ at steady state, all the terms in the expression for $c(t)$ will vanish except the last two terms. Thus at steady state, the response becomes

$$c_{ss}(t) = K e^{-j\omega t} + K^* e^{j\omega t}$$

Regardless of whether there are simple or multiple poles of $G(s)$, the contribution due to them to the steady state response will zero.

Where the constant K can be evaluated as follows:

$$K = G(s) \frac{A\omega}{(s^2 + \omega^2)(s + j\omega)} \Big|_{s=-j\omega} = -\frac{AG(-j\omega)}{2j}$$

$$K^* = G(s) \frac{A\omega}{(s^2 + \omega^2)(s + j\omega)} \Big|_{s=j\omega} = \frac{AG(j\omega)}{2j}$$

Since $G(j\omega)$ is a complex quantity, it can be written in the form $G(j\omega) = |G(j\omega)| e^{j\theta(\omega)} = M e^{j\theta(\omega)}$,

Where $M = |G(j\omega)|$ represents the magnitude and θ represents the angle of $G(j\omega)$.

$$\theta = \angle G(j\omega) = \tan^{-1} \left[\frac{\text{Imaginary part of } G(j\omega)}{\text{Real part of } G(j\omega)} \right]$$

Similarly, $G(-j\omega) = |G(-j\omega)| e^{-j\theta(\omega)} = |G(j\omega)| e^{-j\theta(\omega)}$.

We can now write,

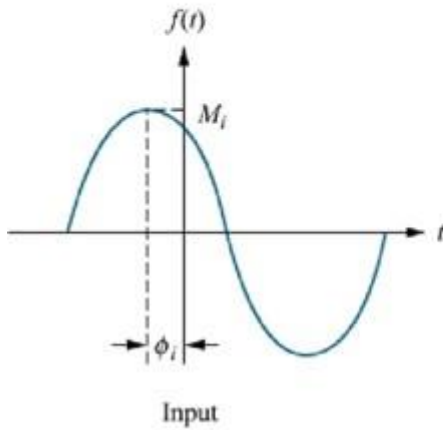
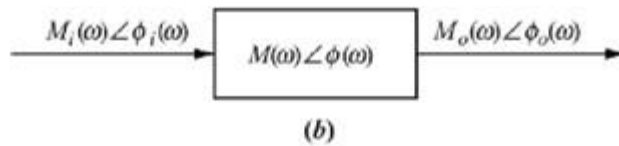
$$c_{ss}(t) = A |G(j\omega)| \frac{e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)}}{2j}$$

$$= A |G(j\omega)| \sin(\omega t + \theta)$$

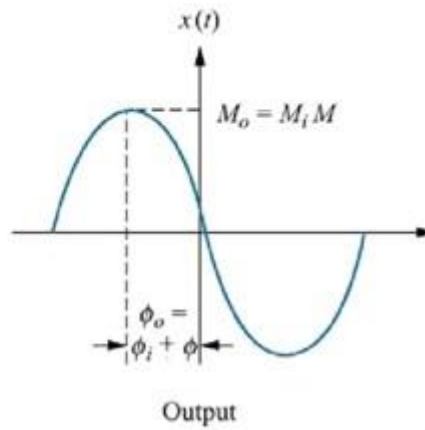
$$= B \sin(\omega t + \theta)$$

Where $B = A |G(j\omega)| = AM$

Hence, for a stable, LTI system, subjected to sinusoidal input, the amplitude of the output is given by the product of that of the input and $|G(j\omega)|$, while the phase angle differs from that of the input by an amount $\theta = \angle G(j\omega)$. A positive phase angle θ is called phase lead where as a negative phase angle is called phase lag.



Input sinusoid



SS output sinusoid

$$G(s) = \frac{K(s + z_1)(s + z_2) \dots (s + z_k)}{s^m (s + p_1)(s + p_2) \dots (s + p_n)}$$

Frequency Domain parameters of prototype 2nd order systems:

The closed loop transfer function of a prototype 2nd order system is given by

Where ζ is the damping factor and ω_n is the undamped natural frequency.

The sinusoidal transfer function of the system is obtained by substituting $s = j\omega$.

Hence,

$$= \frac{1}{(1 - u^2) + j2\zeta u}$$

Where, $u = \omega / \omega_n$ is the normalized driving signal frequency

$$M = |T(j\omega)| = 1 / \sqrt{[(1 - u^2)^2 + (2\zeta u)^2]}$$

$$\angle T(j\omega) = \phi = - \tan^{-1} \left[\frac{2\zeta u}{(1 - u^2)} \right]$$

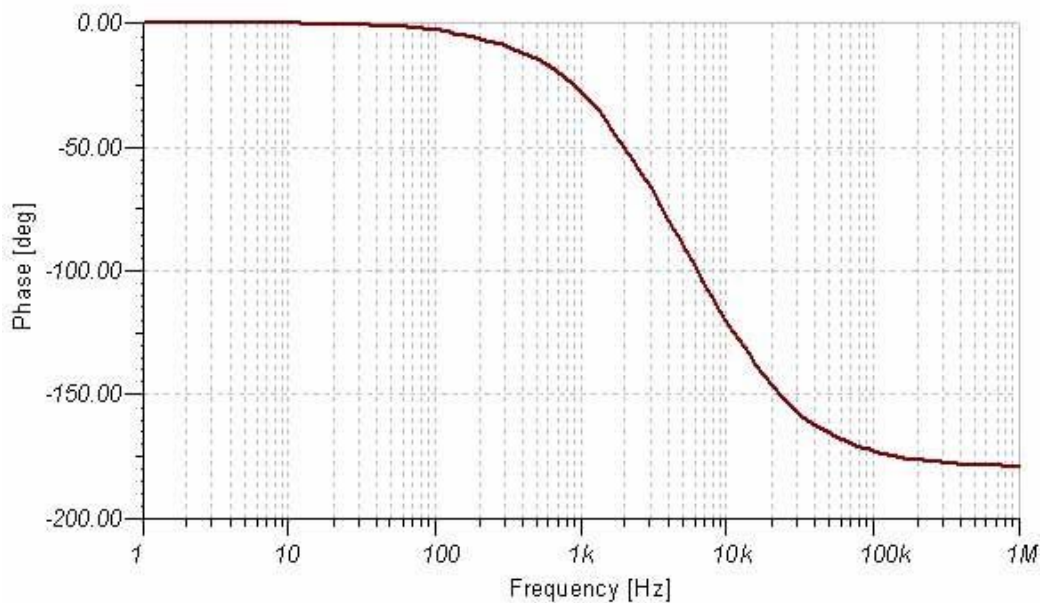
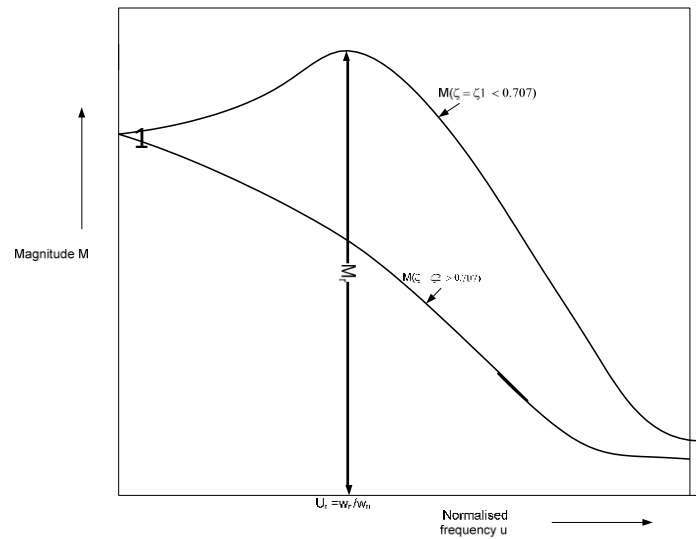
It is seen that when

$$u = 0, M = 1 \text{ and } \phi = 0$$

$$u = 1, M = 1 / 2\zeta \text{ and } \phi = -\pi / 2$$

$$\omega \rightarrow \infty, M \rightarrow 0 \text{ and } \phi \rightarrow -\pi$$

The Magnitude and Phase angle characteristics for normalized frequency ω for certain values of ζ are shown below.



The frequency where M has a peak value is known as Resonant Frequency. At this frequency, slope of the Magnitude curve is zero. Let ω_r be the resonant frequency and $(u_r = \omega_r/\omega_n)$ be the normalized resonant frequency. Then

$$\left(\frac{dM}{du}\right)_{(u = u_r)} = -\frac{(1)}{2} \frac{(-4(1 - \zeta^2)u_r + 8(\zeta^2 u_r^3))}{[(1 - \zeta^2 u_r^2)^2 + (2\zeta^2 u_r)^2]^{3/2}} = 0$$

$$(4u_r^3 - 4u_r + 8\zeta^2 u_r^3) = 0$$

$$u_r = \sqrt{1 - 2\zeta^2}$$

$$\text{or, } \omega_r = \omega_n \sqrt{1 - 2\xi^2} \quad (1)$$

The maximum value of the magnitude, known as the Resonant Peak is given by

$$(2)$$

The phase angle θ of $T(j\omega)$ at the resonant frequency is given by

$$\tan^{-1} \left(\frac{\sqrt{1 - 2\xi^2}}{2\xi} \right)$$

From Eqn. (1) and Eqn. (2), it is seen that as ξ approaches zero, ω_r approaches ω_n and M_r approaches infinity. For $0 < \xi \leq 1/\sqrt{2}$, the resonant frequency always has a value less than ω_n and the resonant peak has a value greater than 1.

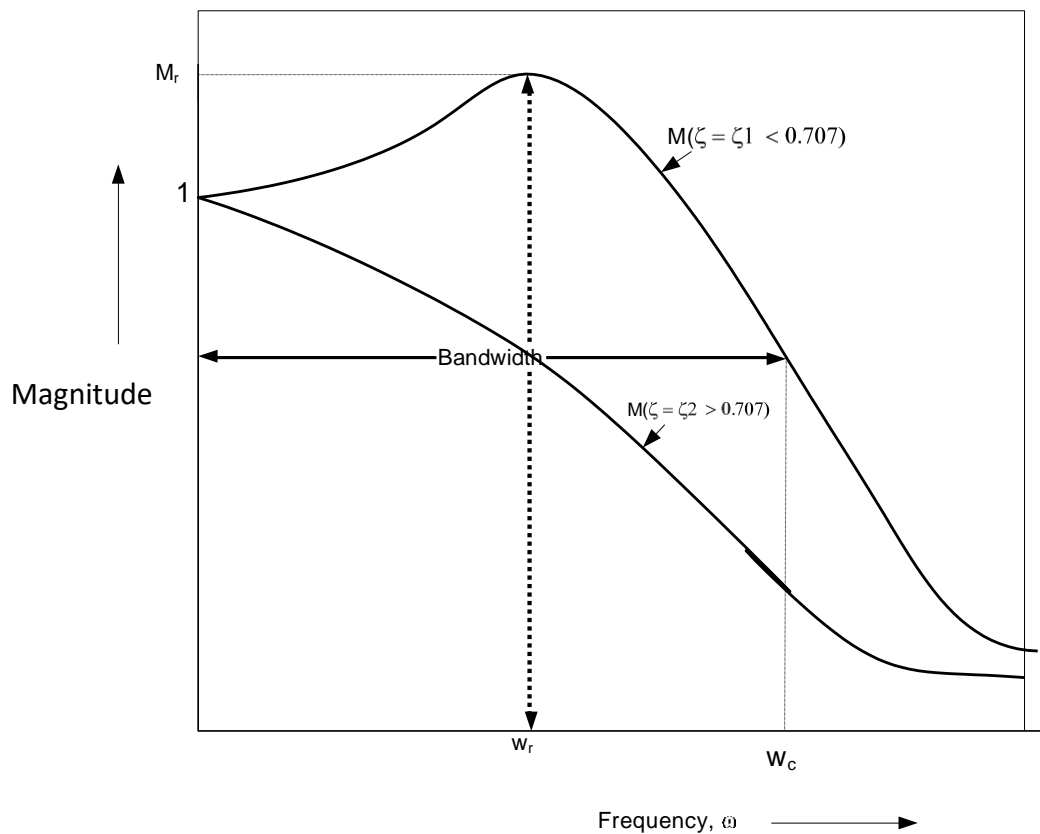
For $\xi > 1/\sqrt{2}$, it is seen that $dM/d\omega$, slope of the magnitude curve does not become zero for any real value of ω . For this range of ξ , the magnitude of M decreases monotonically from $M=1$ at $\omega=0$ with increasing ω , as shown in the above figure. It therefore follows that for $\xi > 1/\sqrt{2}$, there is no resonant peak and as such the greatest value of M equals 1.

As is evident from the above equations, for a second order system, the resonant peak M_r of its frequency response is indicative of its damping factor ξ for $0 < \xi \leq 1/\sqrt{2}$, and the resonant frequency ω_r of the frequency response is indicative of its natural frequency for a given ξ and hence indicative of its speed of response (as $t_{1\%} \approx 4 / (\zeta \omega_n)$). M_r and ω_r of the frequency response could thus be used as performance indices for a second order system.

For $\omega \gg \omega_n$, M decreases monotonically. The frequency at which M has a value $1/\sqrt{2}$ of special significance and is called the cut-off frequency ω_c . The signal frequencies above the cut-off frequency are greatly attenuated in passing through a system.

For feedback control systems, the range of frequencies over which M is equal to or greater than $1/\sqrt{2}$ is defined as the bandwidth ω_b . Control systems being low-pass filters (At zero frequency, $M=1$), the bandwidth is equal to the cut-off frequency ω_c .

In general, the bandwidth of a control system indicates the noise-filtering characteristic of the system. Also, the bandwidth gives a measure of the transient response properties as observed below.



The normalized bandwidth $u_b = \omega_b / \omega_n$ of the second order system under consideration can be determined as follows:

$$M = \frac{1}{\sqrt{[(1 - u_b^2)]^2 + (2 \zeta u_b)^2}} = \frac{1}{\sqrt{2}}$$

$$u_b^4 - 2(1 - 2\zeta^2)u_b^2 + 1 = 0$$

Solving for u_b , we get,

$$u_b = [1 - 2(\zeta^2 + \sqrt{(\zeta^2 - 4(\zeta^2 + 4)})}]^{1/2}$$

It can be approximated in linear form as

$$u_b = -1.19\zeta + 1.85$$

We thus observe that the normalized bandwidth is a function of damping only. The de-normalized bandwidth can be written as

$$\omega_b = \omega_n [1 - 2(\zeta^2 + \sqrt{(\zeta^2 - 4(\zeta^2 + 4)})}]^{1/2}$$

Correlation between Time Domain and Frequency Domain:

Let us consider the step response of the second order system. The peak overshoot M_p of the step response for $0 < \zeta \leq 1$ is

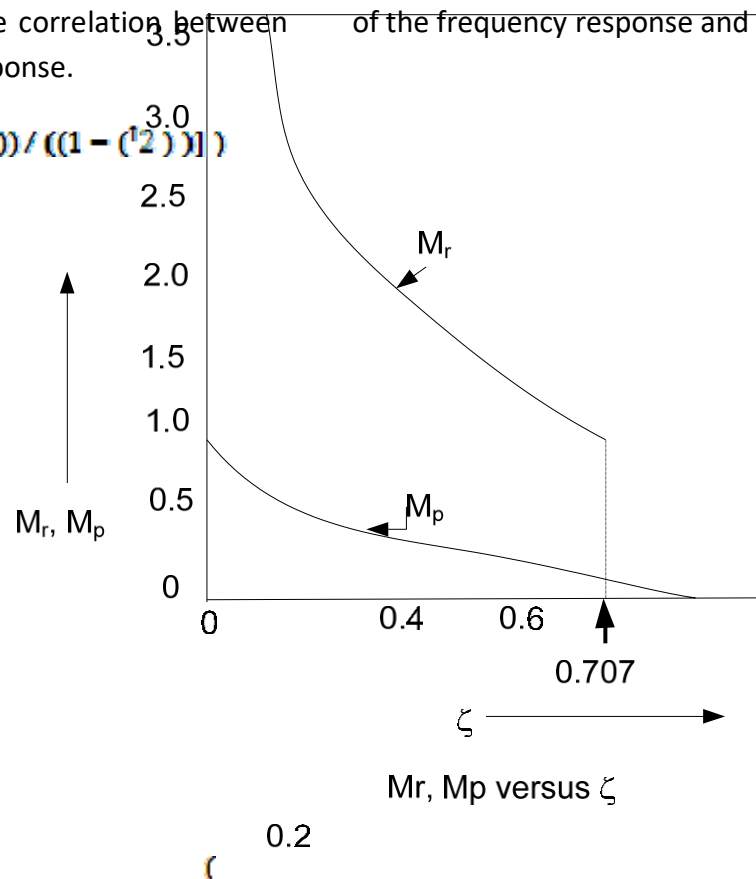
$$M_p = e^{\pi \xi / \sqrt{1 - \xi^2}}$$

The comparison of M_r and M_p plots is shown below. It shows that for $0 < \xi \leq 1/\sqrt{2}$, the two performance indices are correlated as both are functions of the system damping factor ξ only. It means that a system with a given value of M_r of its frequency response, must exhibit a corresponding value of M_p if subjected to a step input. For $\xi > 1/\sqrt{2}$, the resonant peak M_r does not exist and the correlation breaks down. Similarly, the expression for damped natural frequency for a second order system is given as

$$\omega_d = \omega_n \sqrt{1 - \xi^2}$$

Thus, there exists definite correlation between M_r and ω_d of the frequency response and damped frequency of oscillation of the step response.

$$\omega_r / \omega_d = \sqrt{\frac{1 - 2\xi^2}{1 - \xi^2}}$$



It is further observed that the bandwidth, a frequency domain concept, is indicative of the un-damped natural frequency of a system for a given ζ , and therefore indicative of the speed of response ($t_s = 4/(\omega_n \zeta)$), a time-domain concept.

Commonly used frequency response analysis Methods:

Commonly used frequency response analysis Methods are:

- Bode plot
- Nyquist plot
- Nichols chart

Bode plot consists of two simultaneous graphs:

- Magnitude in dB $[(20 \log |G(j\omega)|)]$ (Base 10) vs. frequency (in $\log \omega$)
- Phase (in degrees) vs. frequency (in $\log \omega$)

In the logarithmic representation, the curves are drawn on semilog paper, using the log scale for frequency and the linear scale for either magnitude (in Decibels) or phase angle (in degrees).

Advantages of Bode Plot:

- Multiplication of Magnitudes can be converted into addition
- A simple method of sketching Bode Plot is based on asymptotic approximations. Such information on straight line asymptotes is sufficient if only rough information on frequency- response characteristics is needed.
- Should the exact curve be desired, corrections can be made easily to these basic asymptotic plots.
- Low frequency response contains sufficient information about the physical characteristics of most of the practical systems.
- Experimental determination of a transfer function is possible through Bode plot analysis.

Bode Diagrams

In Bode diagrams, frequency ratios are expressed in terms of:

- **Octave:** it is a frequency band from ω_1 to $2\omega_1$.
- **Decade:** it is a frequency band from ω_1 to $10\omega_1$, where ω_1 is any frequency value.

The basic factors which occur frequently in an arbitrary transfer function are:

- Gain K
- Integral and derivatives: $(j\omega)^{\pm 1}$
- First order factors: $(1 + j\omega T)^{\pm 1}$, $T = \frac{1}{\omega_c}$
- Quadratic Factors: $(1 + 2\zeta(j\omega\omega_n) + (j\omega\omega_n)^2)^{\pm 1}$

Bode Diagrams

- For Constant Gain K, log-magnitude curve is a horizontal straight line at the magnitude of $(20 \log K)$ dB and phase angle is 0 deg.
- Varying the gain K, raises or lowers the log-magnitude curve of the transfer function by the

corresponding constant amount, but has no effect on the phase curve

- Logarithmic representation of the frequency-response curve of factor $(j(\omega/a) + 1)$ can be approximated by two straight-line asymptotes
- Frequency at which the two asymptotes meet is called the **corner frequency** or **break frequency**.

The Gain K:

Magnitude Response:

$$\text{Log Magnitude} = 20 \log K$$

As a number increases by a factor of 10, the corresponding value increases by a factor of 20. This may be seen from the following:

$$20 \log(K \times 10) = 20 \log K + 20$$

$$20 \log(K \times 10^n) = 20 \log K + 20n$$

Again, when expressed in decibels, the reciprocal of a number differs from its value only in sign, i.e., for the number K,

$$20 \log K = -20 \log \frac{1}{K}$$

Integral and Derivative Factors $Q(\omega) = \pm 1$

Log Magnitude Plot:

Log Magnitude of $\frac{1}{j\omega}$ is

$$20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega$$

Phase Plot:

The phase angle of $\frac{1}{j\omega}$ is constant and equal to -90° .

If the log magnitude $-20 \log \omega$ is plotted on a logarithmic scale, it is a straight line. To draw this straight line, we need to locate one point (0 dB, $\omega = 1$) on it. Since

$$-20 \log 10\omega \text{ dB} = (-20 \log \omega - 20) \text{ dB},$$

The slope of the line is -20 dB/decade or -6 dB/octave.

Similarly,

Log Magnitude of $j\omega$ is
 $= 20 \log \omega$

The phase angle of $\frac{1}{j\omega}$ is constant and equal to 90° .

It can be seen that the differences in the frequency responses of $\frac{1}{j\omega}$ and $j\omega$ lie in the slopes of the log-magnitude curves and in the signs of the phase angles.

If the transfer function contains the factor $\left(\frac{1}{j\omega}\right)^n$ or $(j\omega)^n$, the log magnitude becomes respectively,

$$20 \log \left| \frac{1}{(j\omega)^n} \right| = -n \times 20 \log j\omega = -20n \log \omega \text{ dB} \quad \text{or}$$

$$20 \log |(j\omega)^n| = n \times 20 \log j\omega = 20n \log \omega \text{ dB}$$

The slopes of the log-magnitude curve for the factors $\frac{1}{(j\omega)^n}$ and $(j\omega)^n$ are thus $-20n$ dB/decade and $20n$ dB/decade respectively. The phase angle of $\frac{1}{(j\omega)^n}$ is equal to $-90^\circ n$ over the entire frequency range, whereas that of $(j\omega)^n$ is $90^\circ n$ over the entire frequency range. The magnitude curve will pass through the point (0 dB, $\omega = 1$)

First-Order Factors $(1 + j\omega T)^{-1}$:

Log-Magnitude Curve:

The log magnitude of the first order factor $\frac{1}{1 + j\omega T}$ is $20 \log \left| \frac{1}{1 + j\omega T} \right| = -20 \log \sqrt{1 + \omega^2 T^2}$ dB

For low frequencies, such that $\omega \ll \frac{1}{T}$, the log magnitude may be approximated by

$$-20 \log \sqrt{1 + \omega^2 T^2} \approx -20 \log 1 = 0 \text{ dB.}$$

Thus, the log magnitude curve at low frequencies is the constant 0-dB line. For high frequencies, such that $\omega \gg \frac{1}{T}$, $-20 \log \sqrt{1 + \omega^2 T^2} \approx -20 \log \omega T = -20 \log \omega - 20 \log T$ dB.

At $\omega = \frac{1}{T}$, the log magnitude equals 0 dB; at $\omega = \frac{10}{T}$, the log magnitude is -20 dB. Thus, the value of $-20 \log \omega T$ decreases by 20 dB for every decade of ω . For $\omega \gg \frac{1}{T}$, the log-magnitude curve is thus a straight line with a slope of -20 dB/decade (or -6 dB/octave).

Our analysis shows that the logarithmic representation of the frequency-response curve for the factor $1/(1 + j\omega T)$ can be approximated by two straight-line asymptotes, one a straight-line at 0 dB for the

frequency range $0 < \omega < \frac{1}{T}$ and the other a straight line with slope -20 dB/decade for the frequency range

$$\frac{1}{T} < \omega < \infty$$

The frequencies at which the two asymptotes meet is called the *Corner Frequency* or the *Break Frequency*.

For the factor $\frac{1}{1 + j\omega T}$, $\omega = \frac{1}{T}$ is the corner frequency. The corner frequency thus divides the frequency-response curve into two regions: The low frequency region and the high frequency region.

Phase Plot:

The exact phase angle ϕ of the factor $1/(1 + j\omega T)$ is $\phi = \tan^{-1} \omega T$

At zero frequency, the phase angle is 0° . At the corner frequency, the phase angle is

$$\phi = -\tan^{-1}(-1) (T/T) = -\tan^{-1}(-1) 1 = -45^\circ$$

At infinite frequency, the phase angle becomes -90° . Since the phase angle is given by an inverse tangent function, it is skew-symmetric about the inflection point at $\phi = -45^\circ$.

Error in the Magnitude curve:

The error in the Magnitude curve caused by the use of asymptotes can be calculated.

Error at a particular frequency = Actual value – Approximate value of the log-magnitude curve at that frequency

The maximum error occurs at the corner frequency

$$\text{Actual value} = -20 \log \sqrt{1+1} = -20 \log 2 = -3.03 \text{ dB}$$

$$\text{Approximate value} = -20 \log 1 = 0 \text{ dB}$$

Thus, error at corner frequency = -3 dB.

The error at one octave below the corner frequency, i.e., at $\omega = \frac{1}{2T}$ is

$$-20 \log \sqrt{1 + \frac{1}{4}} + 20 \log 1 = -20 \log \left[\frac{\sqrt{5}}{2} \right] = -0.97 \text{ dB}$$

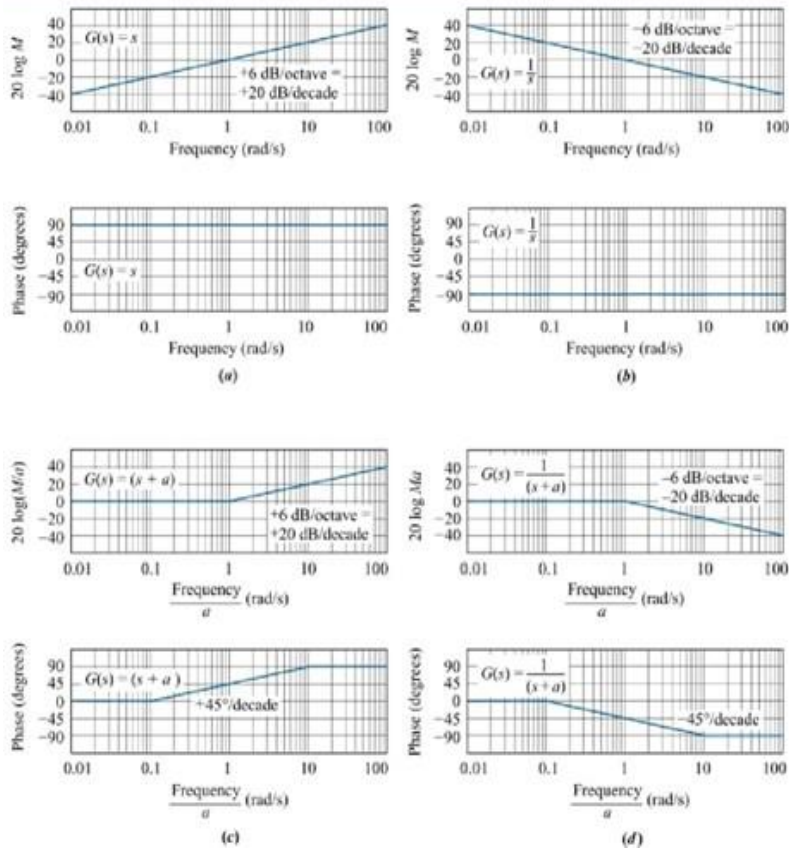
The error at one octave above the corner frequency, i.e., at $\omega = \frac{2}{T}$ is

$$-20 \log \sqrt{2^2 + 1} + 20 \log 2 = -20 \log \left[\frac{\sqrt{5}}{2} \right] = -0.97 \text{ dB}$$

Thus, the error at one octave above or below the corner frequency is approximately -1 dB.

The transfer function $\frac{1}{1 + j\omega T}$ has the characteristics of a low-pass filter. For frequencies above $\omega = \frac{1}{T}$, the log-magnitude falls rapidly towards $-\infty$.

Bode diagrams of some standard first order terms



Ref: N. S. Nise,
Control Systems
Engineering, 4th Ed.
Wiley, 2004.

Bode plots for:

- $G(s) = s$
- $G(s) = 1/s$
- $G(s) = (s + a)$
- $G(s) = 1/(s + a)$

Quadratic Factors $\left[1 + 2\zeta \left(j \frac{\omega}{\omega_n} \right) + \left(j \frac{\omega}{\omega_n} \right)^2 \right]^{-1}$:

When there are complex conjugate zeroes, the prototype 2nd order systems will have the transfer function

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2, \quad 0 < \zeta < 1$$

When there are complex conjugate poles, the prototype 2nd order systems will have the transfer function

$$G(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \quad 0 < \zeta < 1$$

For the complex conjugate poles,

$$G(j\omega) = \frac{1}{\left[1 + 2\zeta \left(j \frac{\omega}{\omega_n} \right) + \left(j \frac{\omega}{\omega_n} \right)^2 \right]}$$

Log Magnitude Curve :

$$\text{Log magnitude} = 20 \log \left| \frac{1}{\left[1 + 2\zeta \left(j \frac{\omega}{\omega_n}\right) + \left(j \frac{\omega}{\omega_n}\right)^2\right]} \right| = -20 \log \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \left(j \frac{\omega}{\omega_n}\right)\right)^2}$$

For low frequencies, i.e., $\ll \omega_n$,

Log magnitude becomes $-20 \log 1 = 0$ dB

The low frequencies asymptote is thus a horizontal line at 0 dB.

For high frequencies i.e., $\gg \omega_n$,

$$\text{Log magnitude becomes } -20 \log \left(\frac{\omega}{\omega_n}\right)^2 = -40 \log \frac{\omega}{\omega_n} \text{ dB} = -40 - 40 \log \frac{\omega}{\omega_n} \text{ dB.}$$

The high frequency asymptote is thus a straight line having the slope -40 dB/ decade.

The high frequency asymptote intersects the low-frequency one at $\omega = \omega_n$, the corner frequency.

The two asymptotes derived are independent of ζ . The resonant peak occurs near the frequency $\omega = \omega_n$.

The damping ratio ζ determines the magnitude of this resonant peak. The magnitude of errors caused by the straight line asymptotes depend on the value of ζ . It is large for small values of ζ .

Phase Plot:

The phase angle of the quadratic factor $\frac{1}{\left[1 + 2\zeta \left(j \frac{\omega}{\omega_n}\right) + \left(j \frac{\omega}{\omega_n}\right)^2\right]}$ is

$$\phi = \angle \frac{1}{\left[1 + 2\zeta \left(j \frac{\omega}{\omega_n}\right) + \left(j \frac{\omega}{\omega_n}\right)^2\right]} = -\tan^{-1} \left[\frac{2\zeta \left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right]$$

The phase angle is a function of both ζ and ω .

$$\begin{aligned} \omega = 0, & \quad \phi = 0^\circ \\ \omega = \omega_n & \quad \phi = -\tan^{-1}(-1) (2\zeta/0) = \tan^{-1}(-1) \infty = -90^\circ \\ \omega = \infty, & \quad \phi = 180^\circ \end{aligned}$$

The phase angle curve is skew-symmetric about the inflection point where $\phi = -90^\circ$.

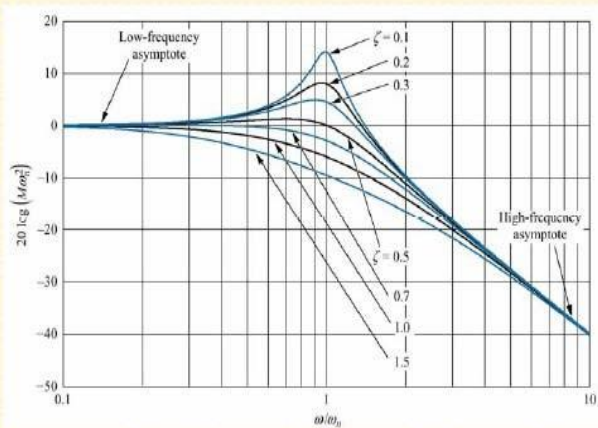
The frequency response for the factor

$$G(j\omega) = \left[1 + 2\zeta \left(j \frac{\omega}{\omega_n}\right) + \left(j \frac{\omega}{\omega_n}\right)^2\right]$$

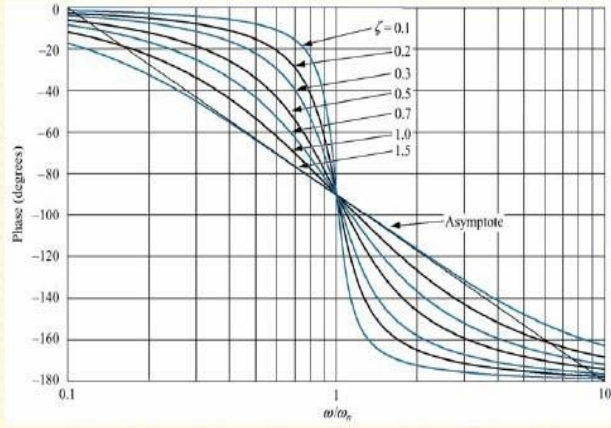
Can be obtained by merely reversing the sign of the log magnitude and that of the phase angle for the factor

$$G(j\omega) = \frac{1}{1 + 2\zeta \left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

Scaled Response for $1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$



Magnitude Plot



Phase Plot

Relationship between System Type and Log-Magnitude Curve:

For a unity feedback system the static position, velocity and acceleration error constants describe the low-frequency behavior of type 0, type 1, and type 2 systems respectively. For a given system, only one of the static error constants is finite and significant. (The larger the value of the finite static error constant, the higher the loop gain is as ω approaches zero.)

The type of the system determines the slope of the log-magnitude curve at low frequencies. Thus, information concerning the existence and magnitude of the steady-state error of a control system to a given input can be determined from the observation of the low-frequency region of the log-magnitude curve.

Determination of Static Error constants:

Assume that the open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K(T_{a_1}s + 1)(T_{b_1}s + 1) \dots (T_{m_1}s + 1)}{s^N(T_{a_2}s + 1)(T_{b_2}s + 1) \dots (T_{p_2}s + 1)}$$

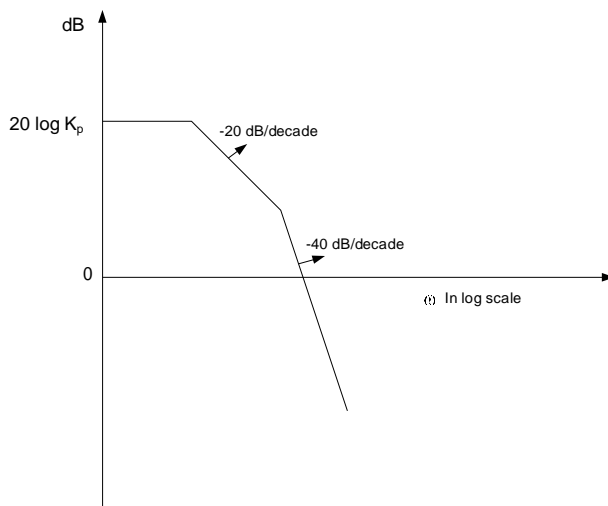
Or

$$G(j\omega) = \frac{K(T_{a_1}j\omega + 1)(T_{b_1}j\omega + 1) \dots (T_{m_1}j\omega + 1)}{(j\omega)^N(T_{a_2}j\omega + 1)(T_{b_2}j\omega + 1) \dots (T_{p_2}j\omega + 1)}$$

Static Position Error constant:

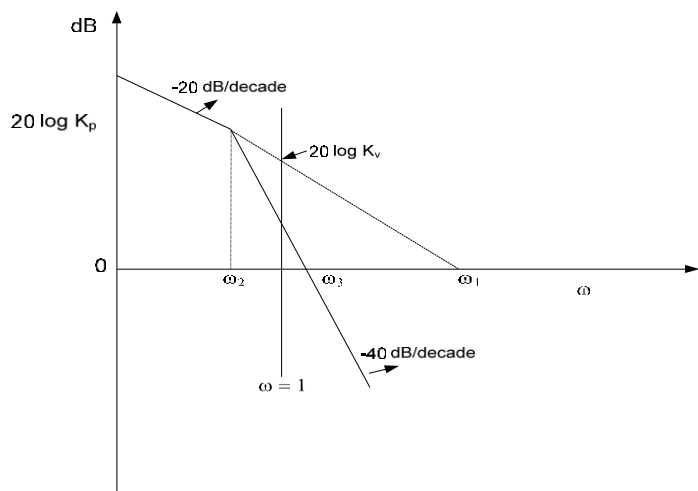
The figure shown below shows an example of the log-magnitude plot of a type 0 system. In such a system, the magnitude of $G(j\omega)$ equals K_p at low frequencies, or

$$\lim_{\omega \rightarrow 0} G(j\omega) = K = K_p$$



Static Velocity Error constant:

The figure given below shows an example of the log-magnitude of a type 1 unity feedback system. The intersection of the initial -20 dB/decade segment (or its extension) with the line $\omega = 1$ has the magnitude $20 \log K_v$. This may be seen as follows.



In a type-1 system,

$$G(j\omega) = \frac{K_v}{j\omega}, \text{ for } \omega \ll 1.$$

Thus,

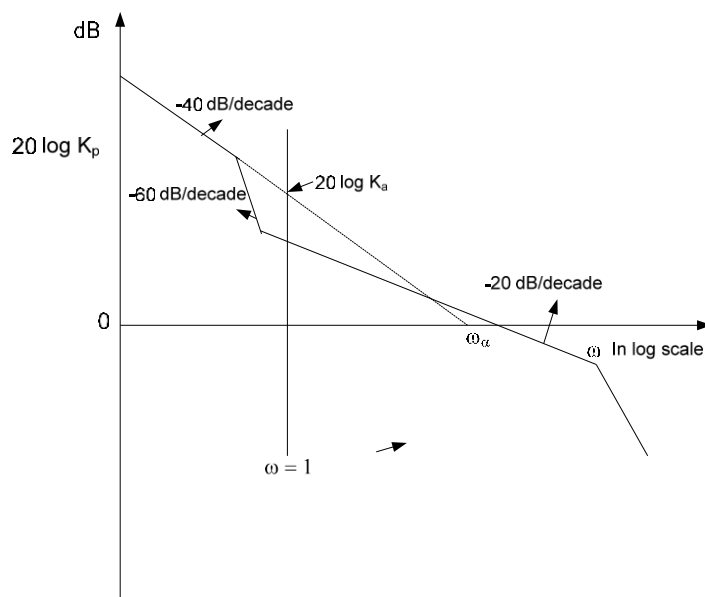
$$20 \log \left(\frac{K_v}{j\omega} \right)_{\omega=1} = 20 \log K_v$$

The intersection of the initial -20 dB/decade segment (or its extension) with the 0-dB line has a frequency numerically equal to K_v , i.e., if the frequency at this intersection is ω_1 , then

$$\left| \frac{K_v}{j\omega} \right| = 1 \quad \text{or,} \quad K_v = \omega_1$$

Static Acceleration Error constant:

The figure given below shows an example of the log-magnitude of a type 2 unity feedback system. The intersection of the initial -40 dB/decade segment (or its extension) with the line $\omega = 1$ has the magnitude $20 \log K_a$. This may be seen as follows.



Since at low frequencies,

$$G(j\omega) = \frac{K_a}{(j\omega)^2}, \text{ for } \omega \ll 1.$$

It follows that

$$20 \log \left(\frac{K_a}{(j\omega)^2} \right)_{\omega=1} = 20 \log K_a$$

The frequency ω_a at the intersection of the initial -40 dB/decade segment (or its extension) with the 0-dB line gives the square root of K_a numerically. This can be seen from the following.

$$20 \log \left| \frac{K_a}{(j\omega_a)^2} \right| = 20 \log 1 = 0$$

Which yields
$$\omega_a = \sqrt{K_a}$$

Phase Margin (PM):

Phase margin is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability.

Gain Crossover Frequency:

Gain crossover frequency is that frequency at which, $|G(j\omega)|$, the magnitude of the open loop transfer function is unity.

The Phase margin PM is 180° plus the phase angle ϕ of the open loop transfer function at the gain crossover frequency.

$$PM = (180^\circ + \phi)_{at\ GCF}$$

Gain Margin (PM):

Gain Margin is the reciprocal of the magnitude $|G(j\omega)|$ at the Phase crossover frequency.

Phase Crossover Frequency:

Phase crossover frequency is that frequency at which, $\angle G(j\omega)$, the phase angle of the open loop transfer function equals -180° .

Thus, Gain Margin,

$$GM = \left(\frac{1}{|G(j\omega)|} \right)_{at\ PCF}$$

In terms of decibels,

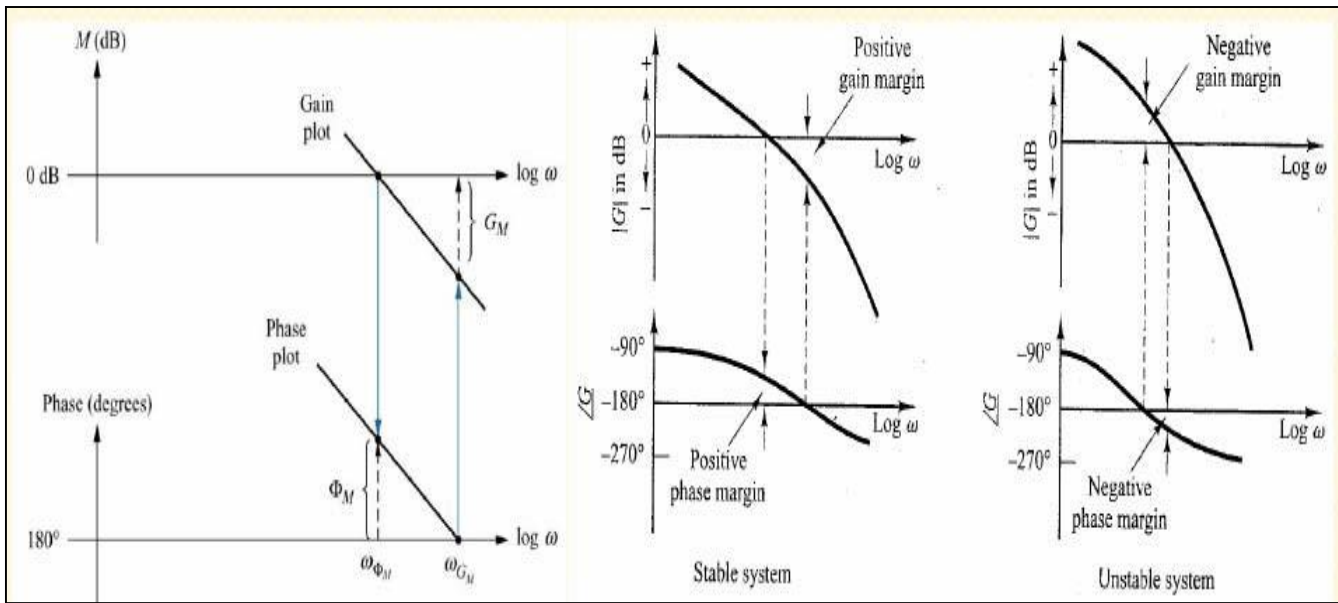
$$GM\ dB = 20 \log |G(j\omega)|_{at\ PCF}$$

A Few Comments on Phase and Gain Margins:

- For a stable non-minimum phase system, the gain margin indicates how much the gain can be increased before the system becomes unstable. For an unstable system, the gain margin indicates how much the gain can be decreased before the system becomes stable.
- The Gain Margin of a first and Second order system is infinite since the polar plot of such systems does not cross the real axis. Thus, theoretically, the 1st and 2nd order systems cannot be unstable.
- It is important to point out that conditionally stable systems will have two or more phase crossover frequencies and some higher order systems with complicated numerator dynamics may also have two or more gain crossover frequencies. For stable systems having two or more gain crossover frequencies, the Phase Margin is measured at the highest Gain Cross-over Frequency.
- Either the Gain Margin alone or the Phase Margin alone does not give a sufficient indication of the relative stability. Both should be given for determination of stability.

- For satisfactory performance, PM should be between 30° and 60° and the GM should be greater than 6 dB.
- The requirement that the PM be between 30° and 60° means that in Bode diagram, the slope of the log-magnitude curve at the gain crossover frequency should be more gradual than -40 dB/decade. In most practical cases, a slope of -20 dB/decade is desirable. If the slope at the gain crossover frequency is -60 dB/decade or steeper, the system is most likely unstable.

PHASE AND GAIN MARGIN THROUGH BODE PLOTS:

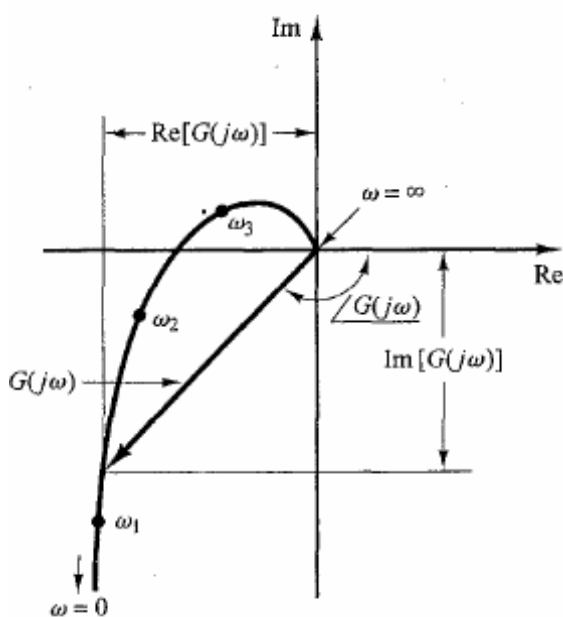


A sinusoidal transfer function $G(j\omega)$ is a complex function and is given by

$$G(j\omega) = \text{Re}[G(j\omega)] + j \text{Im}[G(j\omega)]$$

Or,
$$G(j\omega) = |G(j\omega)| \angle G(j\omega) = M \angle \phi$$

It is seen that $G(j\omega)$ can be represented as a phasor of magnitude M and phase angle (Measured positively in counter-clockwise direction). As the input frequency is varied from 0 to ∞ , the magnitude M and the phase angle change and hence the tip of the phasor $G(j\omega)$ traces a locus in the complex plane. The locus thus obtained is known as 'Polar Plot' as shown below.



Procedure for Sketching of the Polar Plot:

To sketch the Polar Plot of a given Open Loop Transfer Function over the entire frequency range,

- Express the given expression for the OLTF in $(1+sT)$ form.
- Substitute $s = j\omega$ in the expression for $G(s)H(s)$ and get $G(j\omega)H(j\omega)$.
- Find out the expressions for $|G(j\omega)H(j\omega)|$ and $\angle G(j\omega)H(j\omega)$.
- Tabulate various values of magnitude and phase angle for different values of ω starting from 0 to ∞ .
- There are usually four key points to be known.
 - (a) The starting of the plot where $\omega = 0$,
 - (b) The end of the plot where $\omega = \infty$.
 - (c) The point where the Polar plot crosses the real axis, i.e., $\text{Im}(G(j\omega)) = 0$
 - (d) The point where the Polar plot crosses the imaginary axis, i.e., $\text{Re}(G(j\omega)) = 0$.

- Fix all points in a polar graph sheet and join the points. (Polar graph sheet has concentric circles and radial lines. The concentric circles represent the magnitude and the radial lines represent the phase angles. In polar sheet, +ve phase angle is measured in ACW from 0° and -ve phase angle is measured in CW from 0°)

Examples:

Polar Plot of $G(s) = \frac{1}{1+sT}$:

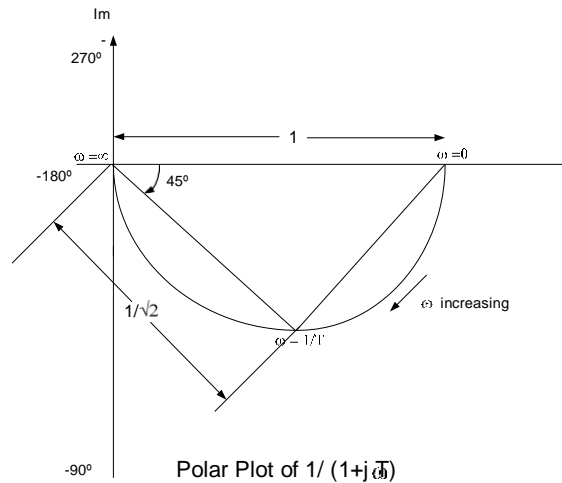
Consider a 1st order system with transfer function $G(s) = \frac{1}{1+sT}$

The sinusoidal transfer function is

$$G(j\omega) = \frac{1}{1+j\omega T}$$

$$(1 + \omega^2 T^2)^{-1/2} \angle -\tan^{-1} \omega T = M \angle \phi$$

When $\omega = 0, M = 1$ and $\phi = 0$. Therefore, the phasor at $\omega = 0$ has unit length and lies along the positive real axis. As ω increases, M decreases and phase angle increases negatively. When $\omega = \frac{1}{T}, M = \frac{1}{\sqrt{2}}$ and $\phi = -45^\circ$. As $\omega \rightarrow \infty$, M becomes zero and ϕ is -90° . This is represented by a phasor of zero length directed along the -90° axis in the complex plane. In fact, the locus of $G(j\omega)$ can be shown to be a semicircle.



Polar Plot of $G(s) = \frac{1}{s(1+sT)}$:

Consider now the transfer function

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$$

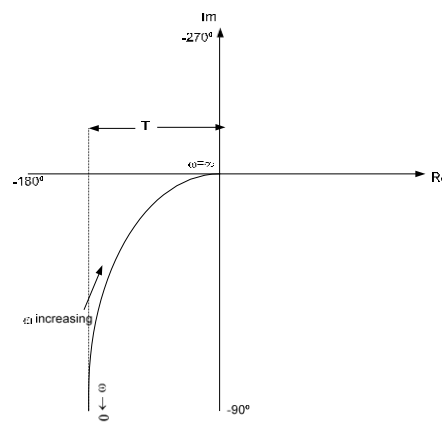
This transfer function may be rearranged as

$$G(j\omega) = \frac{-T}{1 + \omega^2 T^2} - j \frac{1}{\omega(1 + \omega^2 T^2)}$$

$$\lim_{\omega \rightarrow 0} G(j\omega) = -T - j\infty = \infty \angle -90^\circ$$

$$\lim_{\omega \rightarrow \infty} G(j\omega) = -0 - j0 = 0 \angle -180^\circ$$

The general shape of this transfer function is shown below. The plot is asymptotic to the vertical line passing through the point $(-T, 0)$.



Polar Plot of $1/j\omega(1+j\omega T)$

Polar Plot of $G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$:

The low and high frequency regions of the polar plot of the following sinusoidal transfer function

$$G(j\omega) = \frac{1}{\left[1 + 2\zeta \left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2\right]}$$

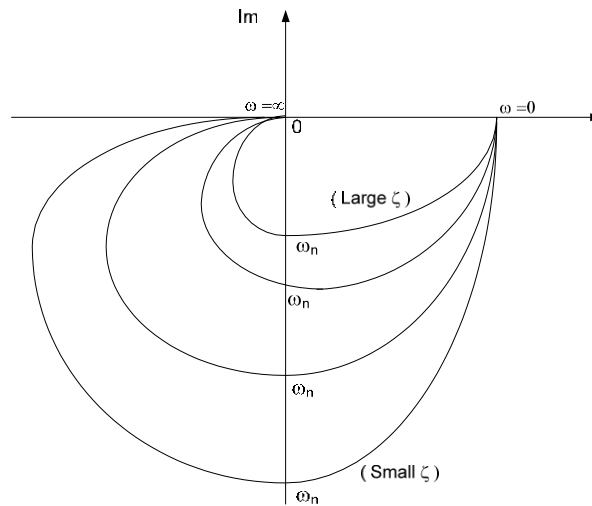
are given respectively by

$$\lim_{\omega \rightarrow 0} G(j\omega) = 1 \angle 0^\circ$$

and

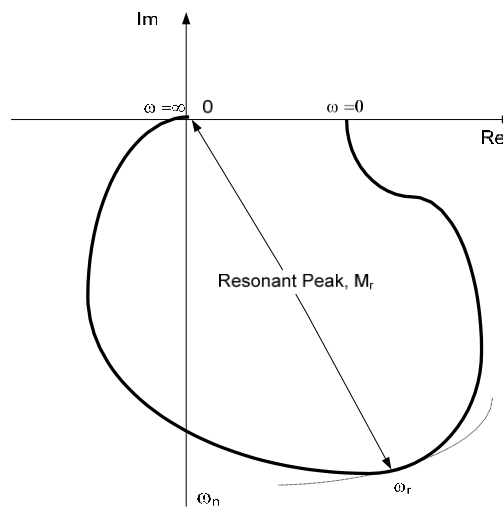
$$\lim_{\omega \rightarrow \infty} G(j\omega) = 0 \angle -180^\circ$$

The Polar plot of this sinusoidal transfer function starts at $1 \angle 0^\circ$ and ends at $0 \angle -180^\circ$ as ω increases from zero to infinity. Thus, the high frequency portion of $G(j\omega)$ is a tangent to the negative real axis.



The exact shape of a polar plot depends on the value of the damping ratio ζ . Thus the general shape is same for both overdamped and underdamped case.

For the under-damped case, at $\omega = \omega_n$, we have $G(j\omega_n) = \frac{1}{2\zeta}$, and the phase angle at $\omega = \omega_n$ is -90° . Therefore, it can be seen that the frequency at which the $G(j\omega)$ locus intersects the imaginary axis is the undamped natural frequency ω_n . The peak value of $G(j\omega)$ is obtained as the ratio of magnitude of the vector at the resonant frequency ω_r to the magnitude of the vector at $\omega = 0$.



For the over-damped case, as ζ increases well beyond unity, the $G(j\omega)$ locus approaches a semi-circle.. This may be seen from the fact that, for a heavily damped system, the characteristic roots are real, and one is much smaller than the other. Since, for sufficiently large ζ , the effect of the larger root (larger in absolute value) on the response becomes very small, the system behaves like a 1st order one.

Example:

Obtain the Polar Plot of the following transfer function:

$$G(j\omega) = \frac{e^{-j\omega L}}{1 + j\omega T}$$

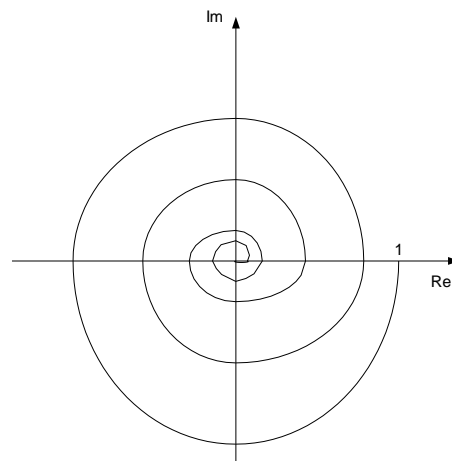
Since $G(j\omega)$ can be written as $(e^{-j\omega L})\left(\frac{1}{1 + j\omega T}\right)$

The magnitude and phase angle are respectively

$$|G(j\omega)| = |e^{-j\omega L}| \left| \frac{1}{1 + j\omega T} \right| = \frac{1}{\sqrt{1 + \omega^2 T^2}}$$

And

$$\angle G(j\omega) = \angle e^{-j\omega L} + \angle \frac{1}{1 + j\omega T} = -\omega L - \tan^{-1} \omega T$$



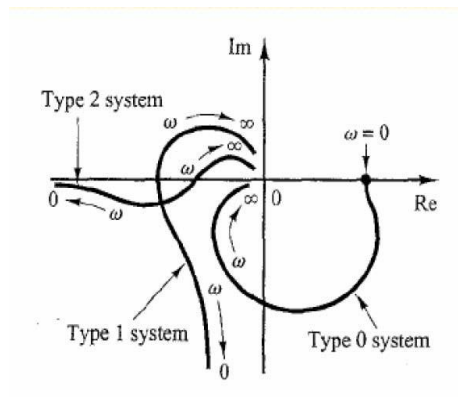
Since the magnitude decreases from unity monotonically, and the phase angle also decreases monotonically, and indefinitely, the polar plot of the given transfer function is a spiral, as shown in the above figure.

General Nature of Nyquist Plots:

The polar plots of a transfer function of the form

$$G(j\omega) = \frac{K(1 + T\omega_1 j\omega)(1 + T\omega_2 j\omega) \dots (1 + T\omega_m j\omega)}{(j\omega)^N (1 + T\omega_1 j\omega)(1 + T\omega_2 j\omega) \dots (1 + T\omega_p j\omega)}$$

Where, $n > m$ or the degree of the denominator polynomial is greater than that of the numerator will have the following shapes.



General shapes of the polar plots of some important functions

The general shapes of the polar plots of some important functions are shown below. From the figures, following observations are made.

Addition of a non-zero pole to a transfer function results in further rotation of the polar plot through an angle of -90° as $\omega \rightarrow \infty$.

- Addition of a pole at the origin to the transfer function rotates the polar plot at zero and infinite frequencies by a further angle of -90° .
- The effect of addition of a zero to the transfer function is to rotate the high frequency portion of the polar plot by 90° in the counter-clockwise direction.
- If degree of the denominator polynomial is greater than that of the numerator, the $G(j\omega)$ loci will converge to the origin clockwise.
- Any complicated shape of the polar plot curves are caused by the numerator dynamics, which is by the time constants in the numerator of the transfer function.

CHAPTER 5, NYQUIST PLOT

Stability in frequency domain:

A stability test for time invariant linear systems can also be derived in the frequency domain. It is known as Nyquist stability criterion.

It is based on the complex analysis result known as Cauchy's principle of argument. Note that the system transfer function is a complex function. By applying Cauchy's principle of argument to the open-loop system transfer function, we will get information about stability of the closed-loop system transfer function and arrive at the Nyquist stability criterion (Nyquist, 1932).

The importance of Nyquist stability lies in the fact that it can also be used to determine the relative degree of system stability by producing the so-called phase and gain stability margins. These stability margins are needed for frequency domain controller design techniques.

We present Only the Essence of the Nyquist stability Criterion and Define the Phase and Gain stability margins. The Nyquist Method is used for studying the stability of linear Systems with Pure time delay.

For a SISO feedback System the closed-loop transfer function is given by:

$$M(s) = \frac{G(s)}{1 + H(s)G(s)}$$

where $G(s)$ represents the system and $H(s)$ is the feedback element.

Since the system poles are determined as those values at which its transfer function becomes infinity, it follows that the closed-loop system poles are obtained by solving the following equation

$$1 + H(s)G(s) = 0 = \Delta(s)$$

which, in fact, represents the System characteristic equation.

In the following we consider the complex function

$$D(s) = 1 + H(s)G(s)$$

Whose zeros are the closed-loop poles of the transfer function. In addition, it is easy to see that the poles of $D(s)$ are the zeros of $M(s)$. At the same time the poles of $D(s)$ are the open-loop control system poles since they are contributed by the poles of $H(s)G(s)$, which can be considered as the open-loop control system transfer function- obtained when the feedback loop is open at some point. The Nyquist stability test is obtained by applying the Cauchy principle of argument to the complex function $D(s)$. First, we state Cauchy's principle of argument.

Cauchy's principle of argument

Let $F(s)$ be an analytic function in a closed region of the complex plane given in Figure below except at a finite number of points (namely, the poles of $F(s)$). It is also assumed that $F(s)$ is analytic at every point on the contour. Then, as s travels around the contour in the s -plane in the clockwise direction, the function $F(s)$ encircles the origin in the $(\text{Re}\{F(s)\}, \text{Im}\{F(s)\})$ -plane in the same direction N times (see Figure 4.6), with N given by

$$N = P - Z$$

Where Z and P stand for the number of zeros and poles (including their multiplicities) of the function $F(s)$ inside the contour.

The above result can be also written as

$$\arg\{F(s)\} = (Z - P)2\pi = 2\pi N$$

Which justifies the terminology used, "the principle of argument".

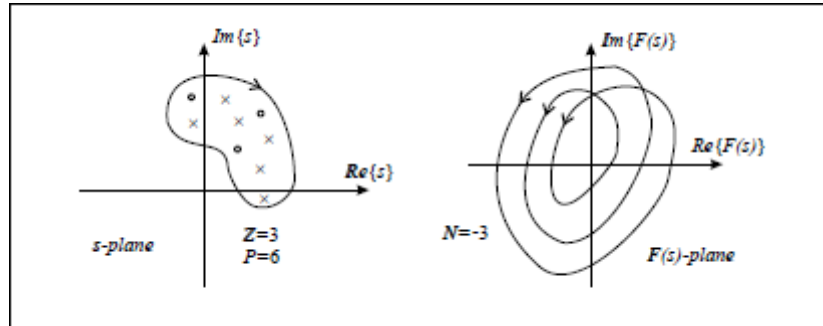


Figure 1. Cauchy's principle of argument

Nyquist Plot

The Nyquist plot is a polar plot of the function $D(s) = 1 + G(s)H(s)$

When s travels around the contour given in Figure below.

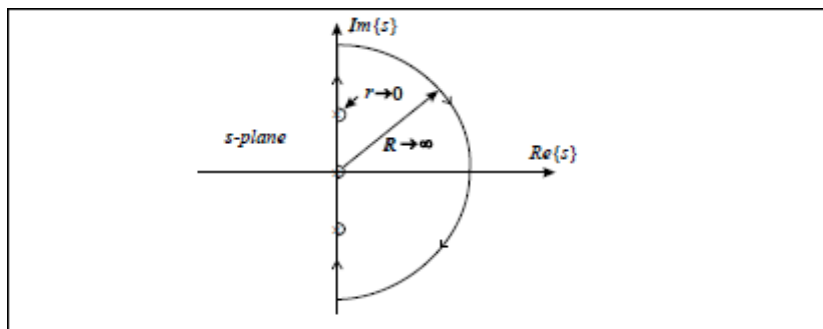


Figure 2. Contour in S- plane

The contour in this figure covers the whole unstable half plane of the complex plane s , $R \rightarrow \infty$. Since the function $D(s)$, according to Cauchy's principle of argument, must be analytic at every point on the contour, the poles of $D(s)$ on the imaginary axis must be encircled by infinitesimally small semicircles.

Nyquist Stability Criterion

It states that the number of unstable closed-loop poles is equal to the number of unstable open-loop poles plus the number of encirclements of the origin of the Nyquist plot of the complex function.

This can be easily justified by applying Cauchy's principle of argument to the function with the s -plane contour given in Figure 2. Note that Z and P represent the numbers of zeros and poles, respectively, of $D(s)$ in the unstable part of the complex plane. At the same time, the zeros of $D(s)$ are the closed-loop system poles, and the poles of $D(s)$ are the open-loop system poles (closed-loop zeros).

The above criterion can be slightly simplified if instead of plotting the function, we plot only the function $G(s)H(s)$ and count encirclement of the Nyquist plot of $G(s)H(s)$ around the point -1 , so that the modified Nyquist criterion has the following form.

Stability via the Nyquist Diagram

We now use the Nyquist diagram to determine a system's stability, using the simple equation. The values of P , the number of open-loop poles of $G(s)H(s)$ enclosed by the contour, and N , the number of encirclements the Nyquist diagram makes about -1 , are used to determine Z , the number of right-half-plane poles of the closed-loop system.

If the closed-loop system has a variable gain in the loop, one question we would like to ask is, "For what range of gain is the system stable?" The general approach is to set the loop gain equal to unity and draw the Nyquist diagram. Since gain is simply a multiplying factor, the effect of the gain is to multiply the resultant by a constant anywhere along the Nyquist diagram.

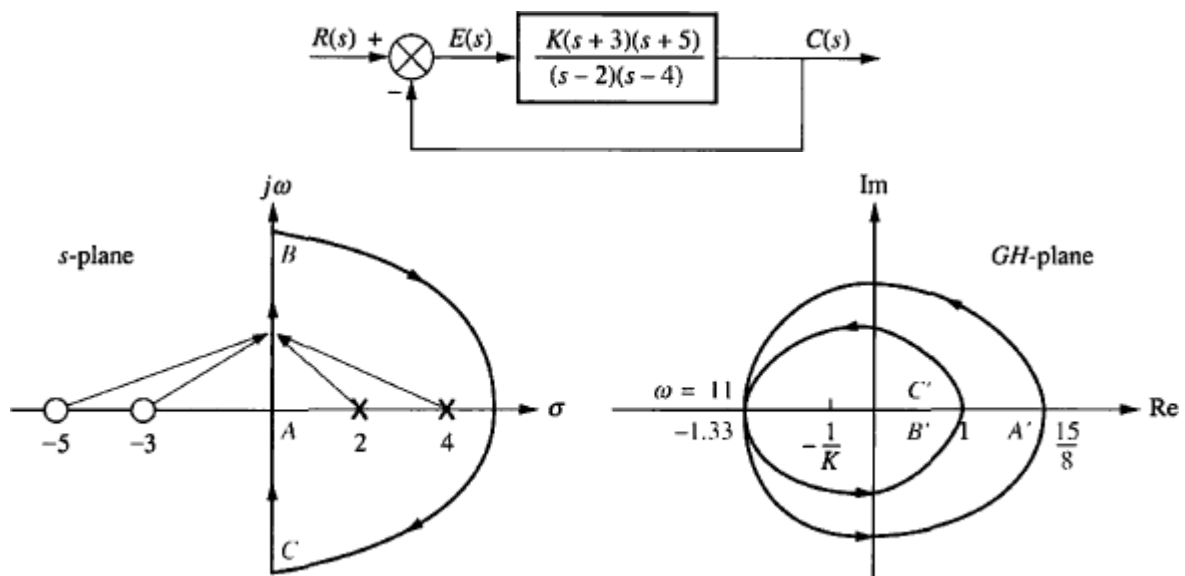


Figure 3. Nyquist stability contour and diagram

As the gain is varied, we can visualize the Nyquist diagram is expanding (increased gain) or shrinking (decreased gain) like a balloon. This motion could move the Nyquist diagram past the -1 point, changing the stability picture. For this system, since $P = 2$, the critical point must be encircled by the Nyquist diagram to yield $N = 2$ and a stable system. A reduction in gain would place the critical point outside the Nyquist diagram where $N = 0$, yielding $Z = 2$, an unstable system.

If the Nyquist diagram intersects the real axis at -1 , then $G(j\omega)H(j\omega) = -1$. From root locus concepts, when $G(s)H(s) = -1$, the variable s is a closed-loop pole of the system. Thus, the frequency at which the Nyquist diagram intersects -1 is the same frequency at which the root locus crosses the $j\omega$ -axis. Hence, the system is marginally stable if the Nyquist diagram intersects the real axis at -1 .

In summary, then, if the open-loop system contains a variable gain, K , set $K = 1$ and sketch the Nyquist diagram. Consider the critical point to be at $-1/K$ rather than at -1 . Adjust the value of K to yield stability, based upon the Nyquist criterion.

PROBLEM: For the unity feedback system, where $G(s) = K/[s(s + 3)(s + 5)]$, find the range of gain, K , for stability, instability, and the value of gain for marginal stability. For marginal stability also find the frequency of oscillation. Use the Nyquist criterion.

SOLUTION: First set $K = 1$ and sketch the Nyquist diagram for the system –

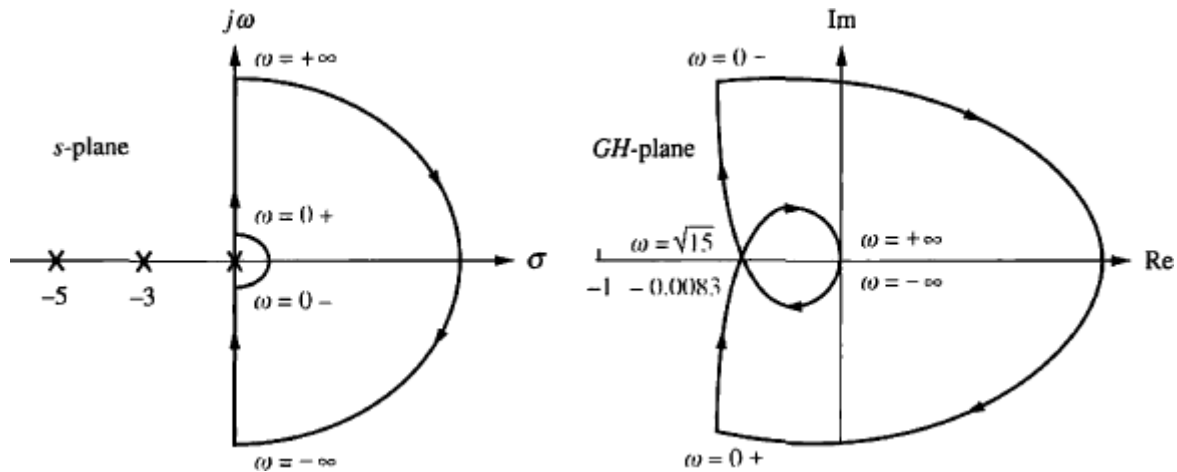


Figure 4. Nyquist contour and stability diagram

For all points on the imaginary axis,

$$G(j\omega)H(j\omega) = \frac{K}{s(s+3)(s+5)} \Big|_{s=j\omega} = \frac{-8\omega^2 - j(15\omega - \omega^3)}{64\omega^4 + \omega^2(15 - \omega^2)^2} \quad (1)$$

At $\omega = 0$, $G(j\omega)H(j\omega) = -0.0356 - j\infty$

Next find the point where the Nyquist diagram intersects the negative real axis. Setting the imaginary part of Eq. (1) equal to zero, we find $\omega = \sqrt{15}$.

Substituting this value of ω back into Eq. (1) yields the real part of -0.0083. Finally, at $\omega = \infty$,

$$G(j\omega)H(j\omega) = G(s)H(s) \Big|_{s=j\omega} = \frac{1}{(j\infty)^3} = 0 \angle (-270^\circ).$$

From the contour of Figure, $P = 0$; for stability N must then be equal to zero. From Figure, the system is stable if the critical point lies outside the contour ($N = 0$), so that $Z = P - N = 0$. Thus, K can be increased by $1/0.0083 = 120.5$ before the Nyquist diagram encircles -1 .

Hence, for stability, $K < 120.5$. For marginal stability $K = 120.5$. At this gain the Nyquist diagram intersects -1 , and the frequency of oscillation is $\sqrt{15}$ rad/s

Stability via Mapping Only the Positive $j\omega$ – Axis

Once the stability of a system is determined by the Nyquist criterion, continued evaluation of the system can be simplified by using just the mapping of the positive $j\omega$ -axis.

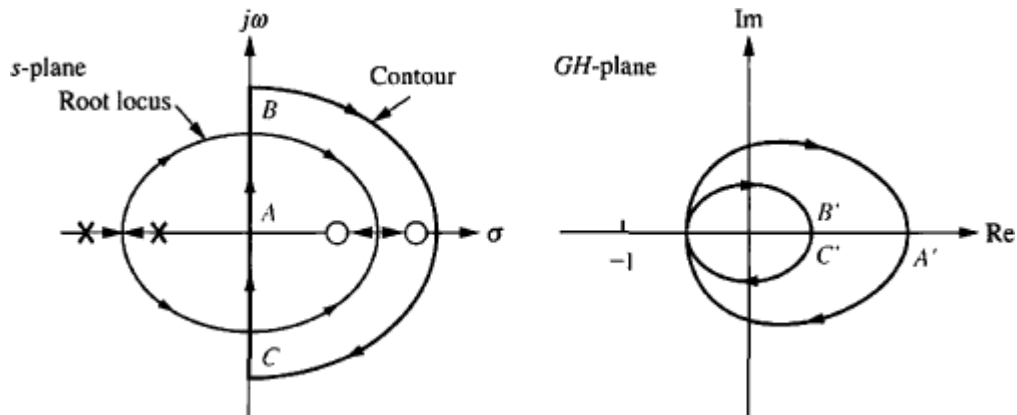


Figure 5. Contour and root locus of system that is stable for small gain and unstable for large gain

Consider the system shown in above Figure, which is stable at low values of gain and unstable at high values of gain. Since the contour does not encircle open-loop poles, the Nyquist criterion tells us that we must have no encirclements of -1 for the system to be stable. We can see from the Nyquist diagram that the encirclements of the critical point can be determined from the mapping of the positive $j\omega$ -axis alone. If the gain is small, the mapping will pass to the right of -1 , and the system will be stable. If the gain is high, the mapping will pass to the left of -1 , and the system will be unstable. Thus, this system is stable for the range of loop gain, K , that ensures that the *open-loop magnitude is less than unity at that frequency where the phase angle is 180° (or, equivalently, -180°)*. This statement is thus an alternative to the Nyquist criterion for this system.

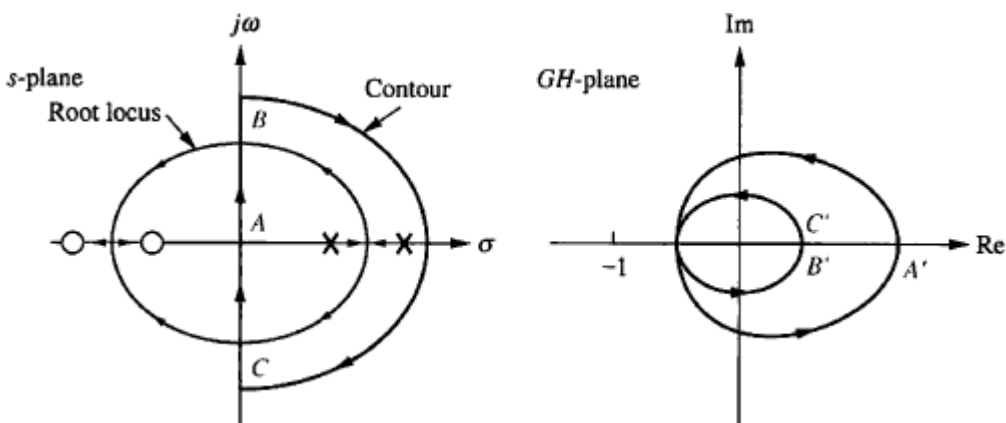


Figure 6. Contour and root locus of system that is unstable for small gain and stable for large gain

Now consider the system shown in above Figure, which is unstable at low values of gain and stable at high values of gain. Since the contour encloses two open-loop poles, two counter clockwise encirclements of the critical point are required for stability. Thus, for this case the system is stable if the *open-loop magnitude is greater than unity at that frequency where the phase angle is 180° (or, equivalently, -180°)*.

In summary, first determine stability from the Nyquist criterion and the Nyquist diagram. Next interpret the

Nyquist criterion and determine whether the mapping of just the positive imaginary axis should have a gain of less than or greater than unity at 180°. If the Nyquist diagram crosses ±180° at multiple frequencies, determine the interpretation from the Nyquist criterion.

PROBLEM: Find the range of gain for stability and instability, and the gain for marginal stability, for the unity feedback system, where $G(s) = K/[(s^2 + 2s + 2)(s + 2)]$. For marginal stability find the radian frequency of oscillation. Use the Nyquist criterion and the mapping of only the positive imaginary axis.

SOLUTION: Since the open-loop poles are only in the left-half-plane, the Nyquist criterion tells us that we want no encirclements of -1 for stability. Hence, a gain less than unity at ±180° is required. Begin by letting $K = 1$ and draw the portion of the contour along the positive imaginary axis as shown in Figure.

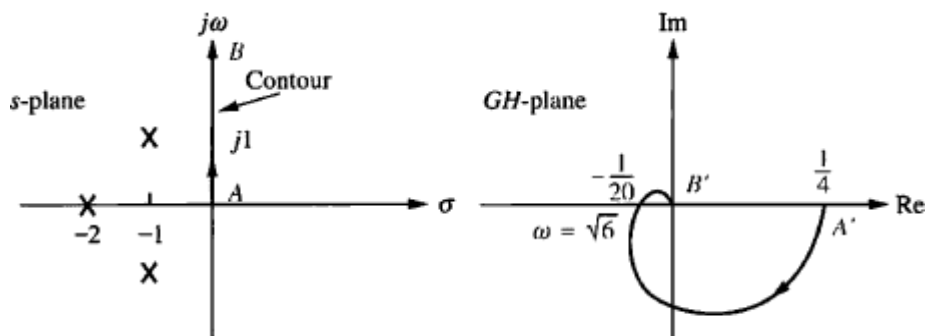


Figure 7. Nyquist diagram of Mapping of positive imaginary axis

In Figure, the intersection with the negative real axis is found by letting $s = j\omega$ in $G(s)H(s)$, setting the imaginary part equal to zero to find the frequency, and then substituting the frequency into the real part of $G(j\omega)H(j\omega)$. Thus, for any point on the positive imaginary axis,

$$G(j\omega)H(j\omega) = \frac{1}{(s^2 + 2s + 2)(s + 2)} \Big|_{s=j\omega}$$

$$= \frac{4(1 - \omega^2)}{16(1 - \omega^2)^2 + \omega^2(6 - \omega^2)^2}$$

Setting the imaginary part equal to zero, we find $\omega = \sqrt{6}$. Substituting this value back into equation yields

$$-\left(\frac{1}{20}\right) = \left(\frac{1}{20}\right) \angle 180^\circ.$$

the real part,

This closed-loop system is stable if the magnitude of the frequency response is less than unity at 180°. Hence, the system is stable for $K < 20$, unstable for $K > 20$, and marginally stable for $K = 20$. When the system is marginally stable, the radian frequency of oscillation is $\sqrt{6}$.

Example:1

Consider the following transfer function

$$G(s) = \frac{K(s + 1)}{s^2(s + 4)(s + 5)}$$

Putting the value of $s = j\omega$ in above equation, we obtain

$$G(j\omega) = \frac{k(j\omega + 1)}{(j\omega)^2(j\omega + 4)(j\omega + 5)}$$

The magnitude and phase angle equations:

$$\frac{k(\sqrt{\omega^2 + 1})}{\omega^2 (\sqrt{\omega^2 + 16})(\sqrt{\omega^2 + 25})}$$

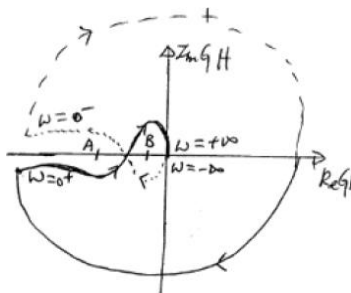
Evaluating magnitude and phase response at $\omega = 0^+$ and $\omega = +\infty$

At $\omega = 0^+$

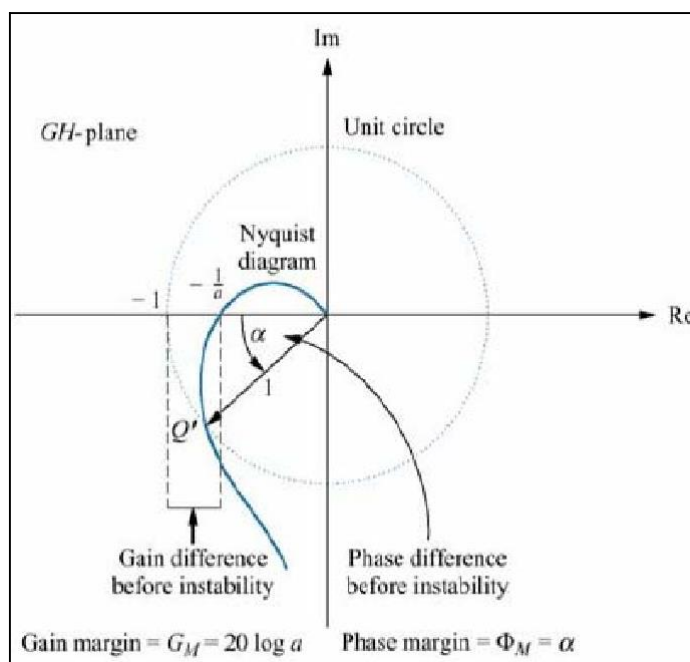
$$|G(j\omega)| \angle G(j\omega) \Rightarrow \infty \angle -180^\circ + \epsilon$$

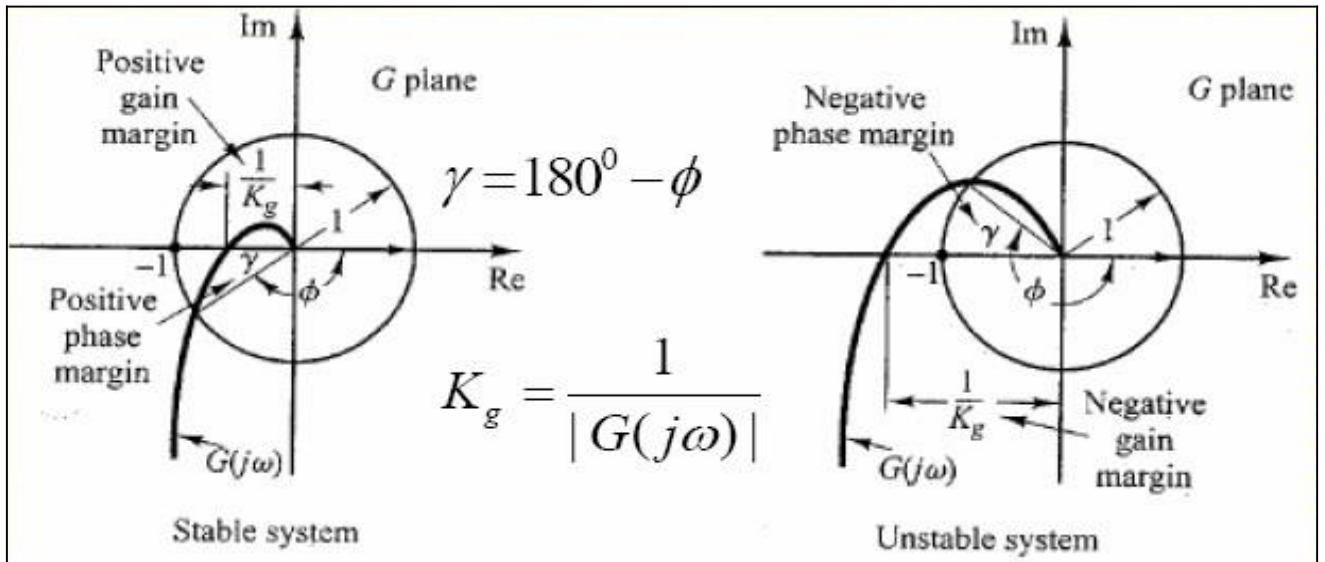
At $\omega = \infty$

$$|G(j\omega)| \angle G(j\omega) \Rightarrow \infty \angle -270^\circ$$



PHASE AND GAIN MARGIN THROUGH NYQUIST PLOTS:





System is said to be

- Stable, if G_M and Φ_M both are positive, i.e. $\omega_{pc} > \omega_{gc}$
- Marginally stable, if G_M and Φ_M both are zero i.e. $\omega_{pc} = \omega_{gc}$
- Unstable, if G_M and Φ_M both are negative i.e. $\omega_{pc} < \omega_{gc}$

The closed-loop frequency response is the locus of the closed-loop magnitude frequency response for unity feedback system. If the frequency response of an open loop system is plotted in polar coordinates, and superimposed on the top of M-circles, then the closed-loop magnitude frequency response is determined by each intersection of this polar plot with the constant M-circles.

M-circles are contours of constant closed-loop magnitude on Nyquist plane.

Let $L(j\omega) = x + jy$. Then $T(j\omega) = \frac{x + jy}{1 + x + jy}$. Hence,

$$|T(j\omega)|^2 = M^2 \Rightarrow M^2(1 + x)^2 + M^2y^2 = x^2 + y^2$$

$$\Rightarrow (1 - M^2)x^2 - 2M^2x + (1 - M^2)y^2 = M^2$$

Then two cases are possible:

$M = 1$ then $x = -\frac{1}{2}$ (vertical line)

$M \neq 1$ then $(1 - M^2)\left(x^2 - 2\frac{M^2}{1 - M^2}x + \frac{M^4}{(1 - M^2)^2} + y^2\right) = M^2$, so we get:

$$\left(x - \frac{M^2}{1 - M^2}\right)^2 + y^2 = \left(\frac{M}{1 - M^2}\right)^2$$

$$\left(x + \frac{M^2}{M^2 - 1}\right)^2 + y^2 = \left(\frac{M}{M^2 - 1}\right)^2$$

Constant M-circles are the circles in the complex plane with radius $M/2\sqrt{|M|^2 - 1}$ centered at

$$\left(-\frac{M^2}{M^2 - 1}, 0\right)$$

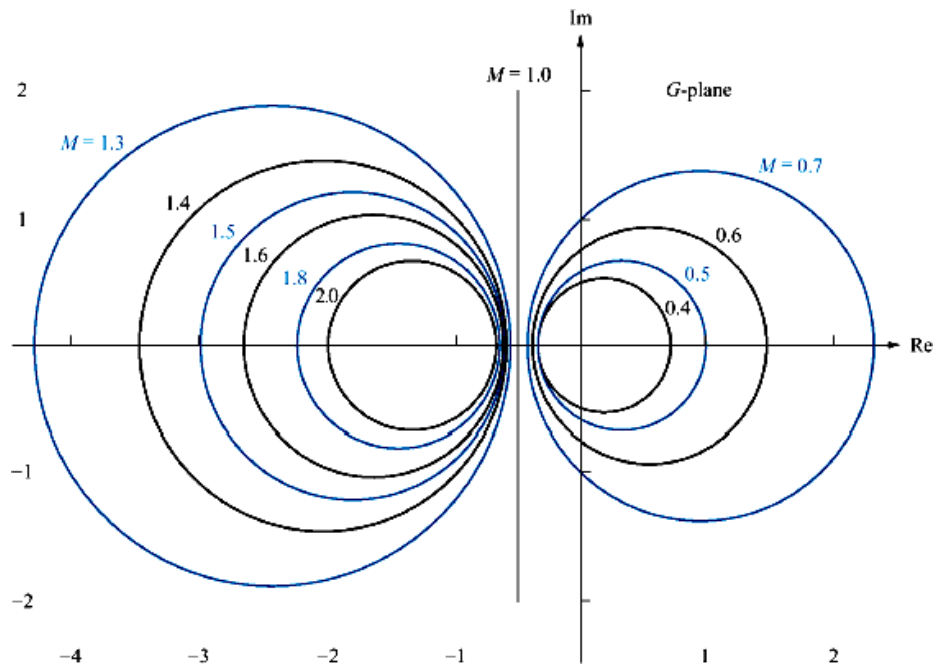


Figure 8. Constant M circles

Constant N-circles

$$\begin{aligned} \angle T(j\omega) = \alpha &= \angle \left(\frac{(x + jy)}{1 + x + jy} \right) \\ &= \tan^{-1} \left(\frac{y}{x} \right) - \tan^{-1} \left(\frac{y}{1 + x} \right) \\ &= \tan^{-1} \left[\frac{y}{x^2 + x + y^2} \right] \end{aligned}$$

Therefore

$$\tan \alpha = \frac{y}{x^2 + x + y^2} = N$$

For a constant value of α , $N = \tan \alpha$ is also constant.

Rearranging the equation we get,

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{N^2 + 1}{(2N)^2}$$

constant N-circles are the circles in the complex plane with radius $\frac{\sqrt{N^2 + 1}}{2N}$ centered at $(-1/2, 1/2N)$ (see figure 2). Constant N-circles are the locus of the closed-loop phase frequency response. Similarly to M-circles, if the frequency response of an open loop system is plotted in polar coordinates, and superimposed on the top of N-circles, then the closed-loop phase frequency response is determined by each intersection of this polar plot with the constant N-circles. All the constant N-circles pass through the origin and $(-1+j0)$ point regardless of the value of N.

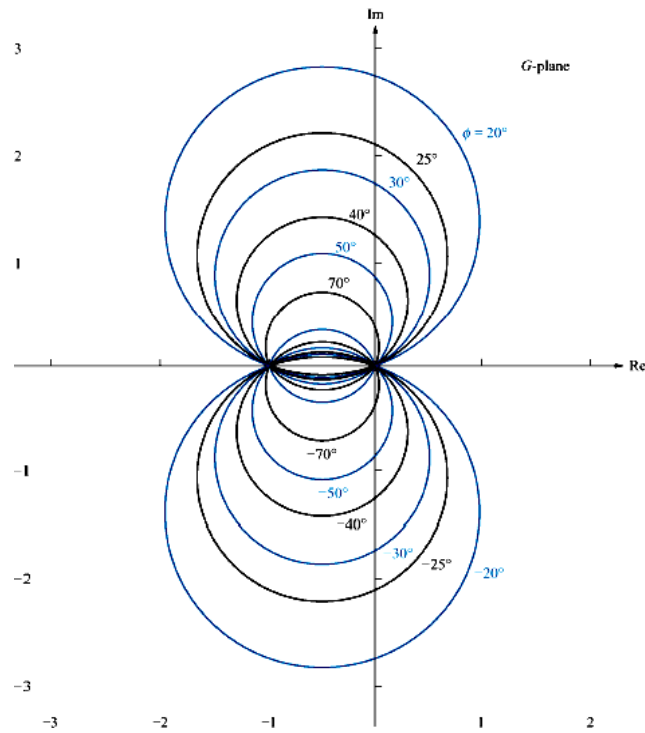


Figure 9. Constant N circles

Example 1- Closed-loop frequency response from open-loop frequency response

Find the closed-loop frequency response of the unity feedback system with open-loop transfer function

$$G(s) = \frac{50}{s(s+3)(s+6)}$$

using the open-loop polar frequency response curve, constant M-circles, and constant N-circles.

Solution

Open-loop frequency response is

$$G(j\omega) = \frac{50}{j\omega(j\omega+3)(j\omega+6)} = \frac{50}{-9\omega^2 + j(18\omega - \omega^3)}$$

Polar plot of $G(j\omega)$ is shown superimposed over the M- and N-circles in figure 3.

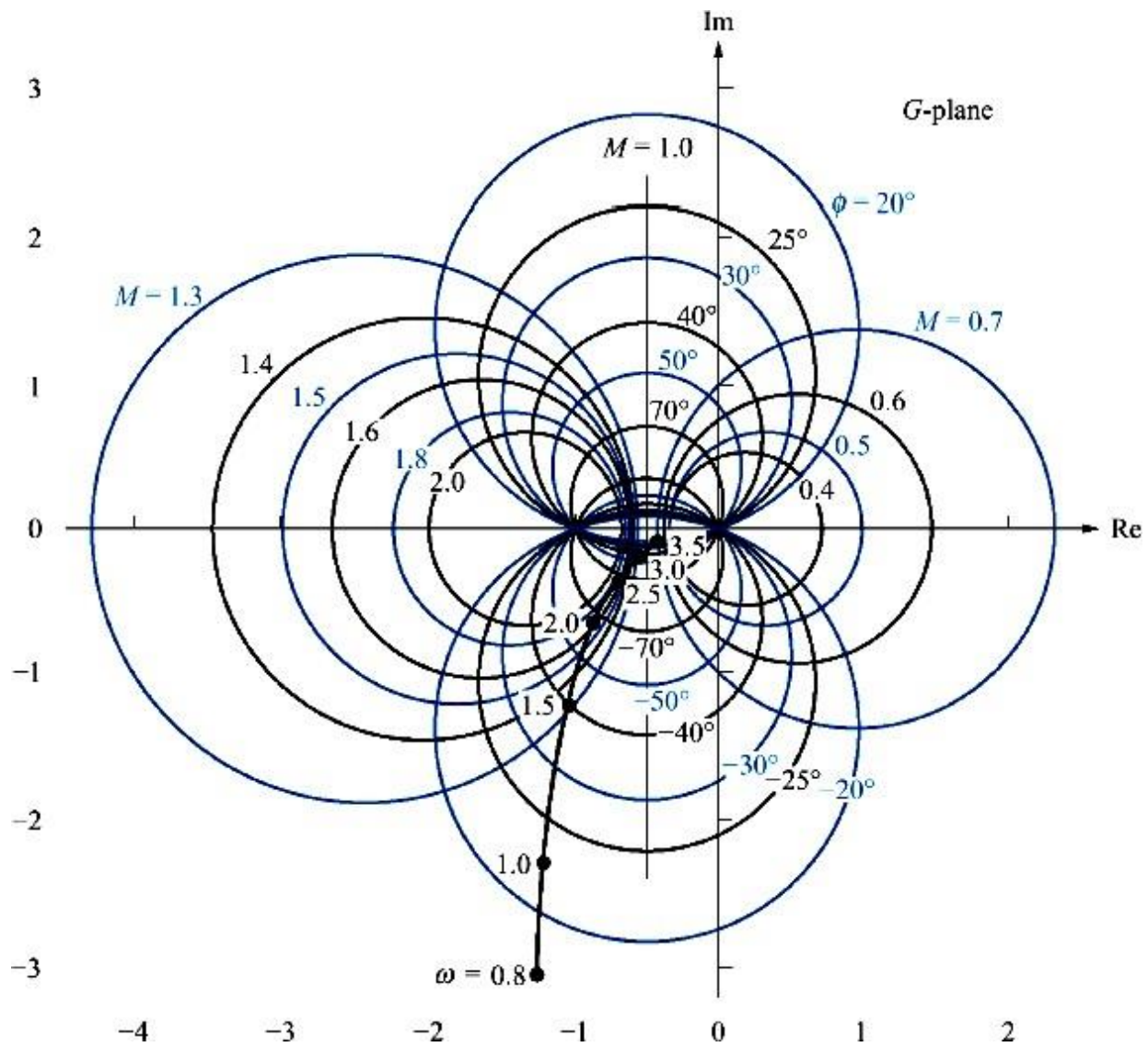


Figure 10. Constant N and M circles

The closed-loop magnitude frequency response can be obtained by finding the intersection of each point of the $G(j\omega)$ with the M-circles, and the closed-loop phase frequency response can be obtained by finding the intersection of each point of the $G(j\omega)$ with the N-circles.

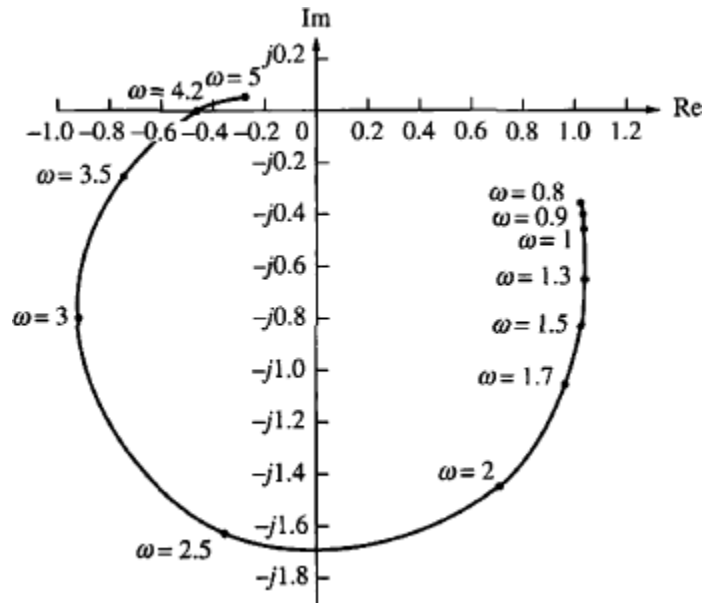
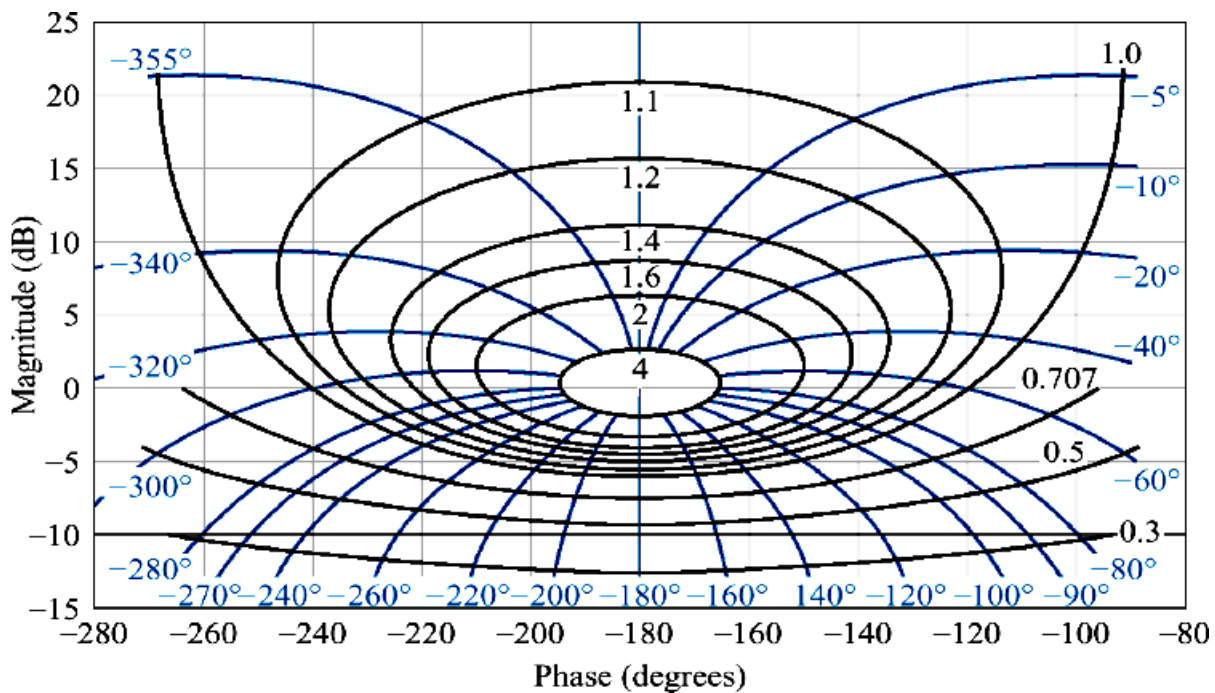


Figure 11. Closed loop frequency response

Nichols Charts

Since it is easier to construct a bode plot than a polar plot, it is preferable to have constant-M and constant- α contours constructed on logarithmic gain and phase coordinates. N.B. Nichols transformed the constant-M and constant- α contours constructed on logarithmic gain and phase coordinate and the resulting chart is known as the Nichols chart. It displays magnitude response in decibels, so that changes in gain are as simple to handle as in the Bode plot. Nichols chart is a plot of open-loop magnitude in dB vs. open-loop phase. Every point on the constant M- and N-circles is transferred to the Nichols chart (see figure 4). The intersection of the $G(j\omega)$ with the Nichols chart yields the frequency response of the closed-loop system.



Example 2- Closed-loop frequency response from open-loop frequency response using Nichols chart
 Consider a unity feedback system with the following open-loop transfer function

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Find the closed-loop frequency response using Nichols chart.

Solution

Superimposing the open-loop frequency response for $K = 1$ on the Nichols chart, we obtain the plot shown in figure 5.

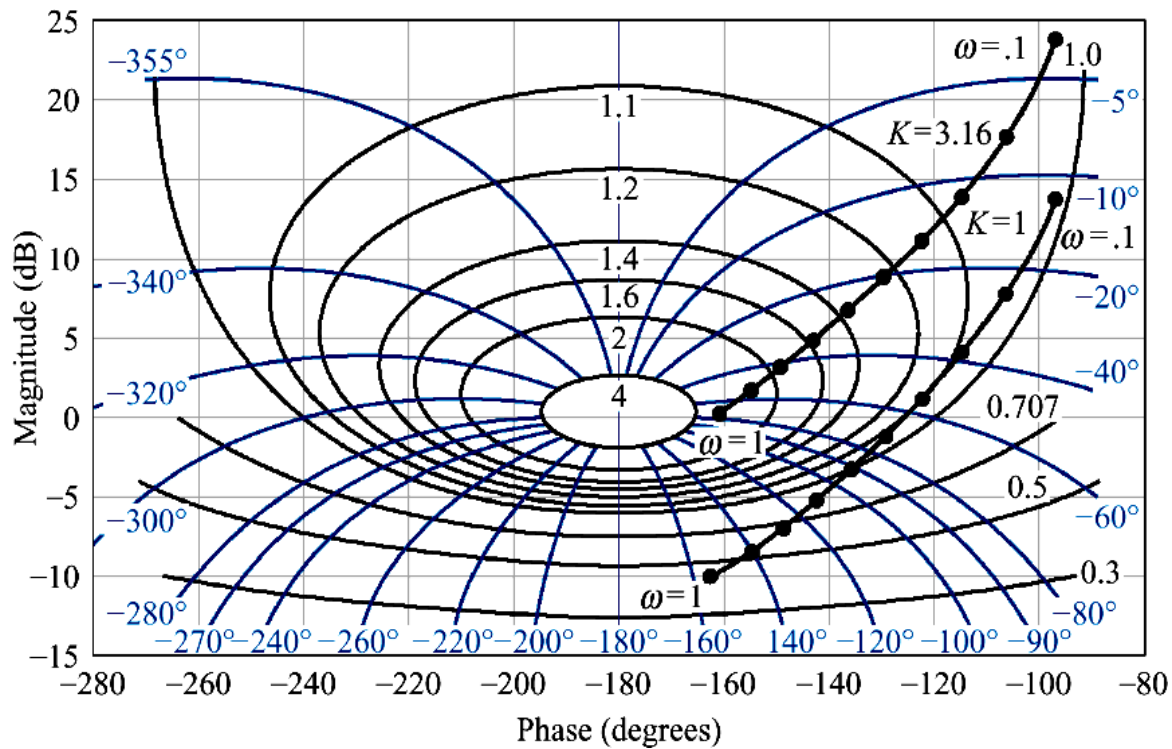


Figure 5: Nichols chart for Example 2

The intersection of the plot of $G(j\omega)$ with the Nichols chart yields the frequency response of the closed-loop system

If the gain is increased by 10 dB, one should simply raise the curve for $K = 1$ by 10 dB to obtain the curve for $K = 3.16$ (10 dB) (see figure 5).

THE END