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UNIT IV

FIR FILTER DESIGN

Introduction

FIR – Finite Impulse Response filters.

→ In many digital applications, FIR filters are preferred over their IIR counterparts.

Advantages of FIR filters

1. FIR filters are always stable
2. FIR filters with exactly linear phase can easily be designed
3. FIR filters can be realized in both recursive & non recursive structures
4. FIR filters are free of limit cycle oscillations when implemented on a finite word length digital system
5. Excellant design methods are available

Disadvantages of FIR filters

1. The implementation of narrow transition band FIR filters are very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders and delay elements
2. Memory Requirement and execution time are very high

Topic 1

Linear Phase FIR Filters

The transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \longrightarrow (1)$$

where h(n) is the impulse response of filter

Fourier transform of h(n) is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \longrightarrow (2)$$

which is periodic in frequency with period 2π

$$H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j\theta(\omega)} \longrightarrow (3)$$

where $|H(e^{j\omega})|$ is magnitude response and $\theta(\omega)$ is phase response.

→ Phase delay and group delay of a filter as

$$\tau_p = \frac{-\theta(\omega)}{\omega} \quad \& \quad \tau_g = \frac{-d\theta(\omega)}{d\omega} \longrightarrow (4)$$

For FIR filters with linear phase and it is defined as

$$\theta(\omega) = -\alpha\omega, \ -\pi \le \omega \le \pi \longrightarrow (5)$$

where α is constant phase delay in samples

The overall Mag response is gn by $\bar{H}(e^{j\omega}) = e^{-j\omega(N-1)/2} \bar{H}(e^{j\omega}) = \bar{H}(e^{j\omega}) e^{j\theta(\omega)} \longrightarrow (a)$

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Sub Eqn (5) in (4) we have $\tau_p = \tau_g = \alpha$, α is independent of frequency $\Rightarrow$

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm |H(e^{j\omega})|\, e^{j\theta(\omega)} \longrightarrow (6)$$

which gives as

$$\sum_{n=0}^{N-1} h(n) \cos\omega n = \pm |H(e^{j\omega})| \cos\theta(\omega) \longrightarrow (7)$$

$$-\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm |H(e^{j\omega})| \sin\theta(\omega) \longrightarrow (8)$$

By taking ratio of Eqn (8) to Eqn (7) $\Rightarrow$

$$\frac{\sum_{n=0}^{N-1} h(n) \sin\omega n}{\sum_{n=0}^{N-1} h(n) \cos\omega n} = \frac{\sin\alpha\omega}{\cos\alpha\omega} \quad [\because \theta\omega = -\alpha\omega] \longrightarrow (9)$$

After simplifying (9), we have

$$\sum_{n=0}^{N-1} h(n) \sin(\alpha - n)\omega = 0$$

$$\sum_{n=0}^{N-1} h(n) \sin(\alpha - n)\omega = 0 \longrightarrow (10)$$

Eqn (10) will be zero when

$$h(n) = h(N-1-n) \longrightarrow (11)$$

and $\alpha = \frac{N-1}{2}$ ⟶ (12)

∴ FIR filters will have constant phase and group delay when the impulse response is symmetrical about $\alpha = \frac{N-1}{2}$

→ The impulse response satisfying Eqn (11) & Eqn (12) for odd and even values of N is shown in Fig 1. N=7 centre of symmetry occurs at 3rd sample, N=6 filter delay is 2½ samples

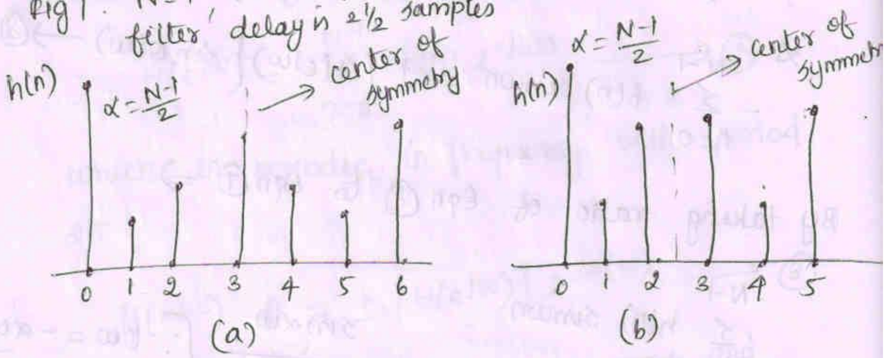

Fig 1: Impulse response sequence of symmetric sequences for a) N odd and b) N even

Example for symmetry: N = 5, h(0) = h(5-1-0) = h(4) h(1)=h(3), h(2)=h(2)

→ If only constant group delay is required and not the phase delay it can be written as

$$\boxed{\theta(\omega) = \beta - \alpha\omega} \longrightarrow (13)$$

$$\Rightarrow \quad H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j(\beta - \alpha\omega)} \longrightarrow (14)$$

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Eqn (14) can be expressed as

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| e^{j(\beta-\alpha\omega)} \rightarrow (15)$$

which gives us

$$\sum_{n=0}^{N-1} h(n) \cos\omega n = \pm |H(e^{j\omega})| \cos(\beta-\alpha\omega) \rightarrow (16)$$

and

$$-\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm |H(e^{j\omega})| \sin(\beta-\alpha\omega) \rightarrow (17)$$

By taking ratio of (17) to (16) $\Rightarrow$

$$\frac{-\sum_{n=0}^{N-1} h(n) \sin\omega n}{\sum_{n=0}^{N-1} h(n) \cos\omega n} = \frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)} \rightarrow (18)$$

This can be obtained as

$$\sum_{n=0}^{N-1} h(n) \sin[\beta-(\alpha-n)\omega] = 0 \rightarrow (19)$$

If $\beta = \pi/2$ Eqn (19) can be written as

$$\sum_{n=0}^{N-1} h(n) \cos(\alpha-n)\omega = 0 \rightarrow (20)$$

Eqn (19) will be satisfied as when

$$h(n) = -h(N-1-n) \longrightarrow (21)$$

and

$$\alpha = \frac{N-1}{2} \longrightarrow (22)$$

$\therefore$ FIR filters have constant group delay τ_g and not constant phase delay when the impulse response is antisymmetrical about $\alpha = \frac{N-1}{2}$ at $N=7$, the centre of antisymmetry occurs at third sample and when $N=6$, the centre of antisymmetry occurs at $2\frac{1}{2}$ samples. It is shown in Fig 2 where $h\left(\frac{N-1}{2}\right) = 0$

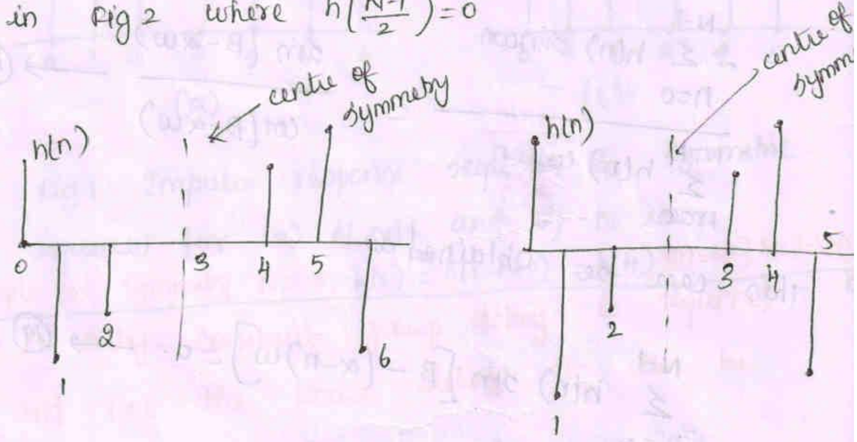

Fig 2: Impulse response for N = odd and N = even proving the antisymmetric

Example for Antisymmetry, $N=6$, 1) $h(0) = -(h(6-1-0)) = -h(5)$

2) $h(1) = -h(4)$

3) $h(2) = -h(3)$

4) $h(3) = -h(2)$, $h(4) = -h(1)$

I.1 CHARACTERISTICS OF LINEAR PHASE FIR FILTERS

Type	Frequency Response $H(e^{j\omega})$	Magnitude Response $\lvert \bar{H}(e^{j\omega}) \rvert$	Phase response $\angle H(e^{j\omega})$	Appln
1) Symmetrical impulse response N odd	$e^{-j\omega(N-1)/2}\left[\sum_{n=0}^{N-1/2} a(n)\cos\omega n\right]$ $a(0) = h\left[\frac{N-1}{2}\right]$ $a(n) = 2h\left\{\left[\frac{N-1}{2}\right] - n\right\}$	$\left\lvert \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n \right\rvert$	$-\alpha\omega + \theta$ $\theta = 0$ for $\bar{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\bar{H}(e^{j\omega}) < 0$	LPF HPF BPF BSF
2) Symmetrical impulse response N even	$e^{-j\omega(N-1)/2}\left[\sum_{n=1}^{N/2} b(n)\cos\left(n-\frac{1}{2}\right)\omega\right]$ $b(n) = 2h\left[\frac{N}{2} - n\right]$	$\left\lvert \sum_{n=1}^{N/2} b(n)\cos\left(n-\frac{1}{2}\right)\omega \right\rvert$	$-\alpha\omega + \theta$ $\theta = 0$ for $\bar{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\bar{H}(e^{j\omega}) < 0$	LPF BPF

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Type	Frequency response $H(e^{j\omega})$	Magnitude response $\lvert\bar{H}(e^{j\omega})\rvert$	Phase Response $\angle H(e^{j\omega})$	Appln
3) Antisymmetrical impulse response N odd	$e^{j\pi/2} e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega n$ $c(n) = 2h\left(\frac{N-1}{2}-n\right)$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega n \right\rvert$	$-\alpha\omega + \pi/2 + \theta$ where $\theta = 0$ for $\bar{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\bar{H}(e^{j\omega}) < 0$	Differentiator, Hilbert Transformer
4) Antisymmetrical impulse response N even	$e^{j\pi/2} e^{-j\omega(N-1)/2} \sum_{n=1}^{N/2} d(n)\sin\omega(n-1/2)$ $d(n) = 2h\left[N/2 - n\right]$	$\left\lvert \sum_{n=1}^{N/2} d(n)\sin\omega(n-1/2) \right\rvert$	$-\alpha\omega + \pi/2 + \theta$ where $\theta = 0$ for $\bar{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\bar{H}(e^{j\omega}) < 0$	Differentiator Hilbert Transformer

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1.2 LOCATION OF ZEROS OF LINEAR PHASE FIR FILTERS

The transfer function of a linear phase FIR filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \longrightarrow ①$$

If $z_0 \neq 0$ is an finite zero of $H(z)$ then

$$H(z)\Big|_{z=z_0} = H(z_0) = \sum_{n=0}^{N-1} h(n) z_0^{-n} = 0$$

$$= h(0) + h(1) z_0^{-1} + \dots + h(N-1) z_0^{-(N-1)} = 0 \quad \to ②$$

For a linear phase filter

$$h(n) = h(N-1-n)$$

$$\therefore H(z)\Big|_{z=z_0} = h(N-1) + h(N-2) z_0^{-1} + \dots h(1) z_0^{-(N-2)} + h(0) z_0^{-(N-1)} = 0$$

$$H(z) = z^{-(N-1)} \left[h(N-1) z_0^{N-1} + h(N-2) z_0^{N-2} + \dots + h(1) z_0^{1} + h(0) \right] = 0$$

$\Rightarrow$ zero locations of linear phase FIR filter is given in fig 1

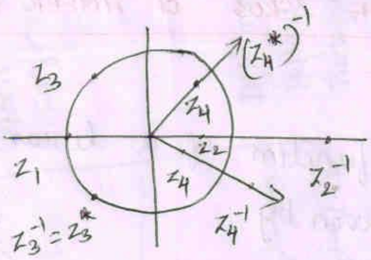

Fig 1: Zero Locations of Linear phase FIR filter

$$= z^{-(N-1)} \sum_{n=0}^{N-1} h(n)\, z_0^{n} = 0$$

$$H(z_0) = z^{-(N-1)} \sum_{n=0}^{N-1} h(n) (z_0^{-1})^{-n} = z^{-(N-1)} H(z_0^{-1}) = 0$$

$$H(z_0) = H(z_0^{-1}) = 0 \longrightarrow (3)$$

From (3) if z_0 is a zero of $H(z)$ then z_0^{-1} is also zero.

<u>Locations of zeros in linear phase FIR filters</u>

1. If $z_1 = -1$ then $z_1^{-1} = z_1$, the zero lies at $z_1 = -1$, this group contains only one zero on the unit circle
2. If z_2 is real zero with $|z_2| < 1$, then z_2^{-1} is also a real zero and there are two zeros in this group

3. If z_3 is a complex zero with $|z_3| = 1$ then $z_3^{-1} = z_3^*$ and there are two zeros in this group

4. If z_4 is a complex zero with $|z_4| \neq 1$. This group contains four zeros $z_4, z_4^{-1}, z_4^*, (z_4^*)^{-1}$

Problems on Linear Phase FIR filters

1. Determine the freq response of FIR filter defined by $y(n) = 0.25x(n) + x(n-1) + 0.25x(n-2)$. Calculate phase delay and group delay

Soln

Gn

$$y(n) = 0.25x(n) + x(n-1) + 0.25x(n-2)$$

Taking Fourier Transform on both sides

$$Y(e^{j\omega}) = 0.25X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega}) + 0.25e^{-2j\omega} X(e^{j\omega}) \rightarrow ①$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 0.25 + e^{-j\omega} + 0.25e^{-2j\omega} \rightarrow ②$$

$$= e^{-j\omega}(0.25e^{j\omega} + 1 + 0.25e^{-j\omega})$$

$$= e^{-j\omega}(1 + 0.5\cos\omega) \rightarrow ③$$

$$= e^{-j\omega}\bar{H}(e^{j\omega}) \rightarrow ④$$

Compare eqn ④ with ⓐ

$$\boxed{H(e^{j\omega}) = e^{-j\omega(N-1)/2}\ \bar{H}(e^{j\omega}) = \bar{H}(e^{j\omega})\, e^{j\theta(\omega)}} \rightarrow (5)$$

we get $\theta(\omega) = -\omega$

The phase delay $\tau_p = \frac{-\theta(\omega)}{\omega} = \frac{\omega}{\omega} = 1$

The group delay $= \frac{-d\theta(\omega)}{d\omega} = \frac{-d}{d\omega}(-\omega) = 1$

HW

2) If the freq response of a linear phase FIR filter is given by

$$H(e^{j\omega}) = e^{-j2\omega}(0.30 + 0.5\cos\omega + 0.3\cos 2\omega)$$

determine filter coefficients

Filter design techniques: Fourier series, windowing & Freq sampling

TOPIC II

II.1) Fourier series Method for Designing FIR filters:

The frequency response $H(e^{j\omega})$ of a system is periodic in 2π. From Fourier series it is known that any periodic function can be expressed as a linear combination of complex exponentials. Desired frequency response of an FIR filter can be represented by the Fourier series.

(11)

II.2) Procedure to FIR filter Design by Fourier Series Method : (i) Choose Desired Freq Response

(i) $$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)\, e^{-j\omega n} \longrightarrow (1)$$

(ii) Impulse/Infinite Filter Sequence: Inverse Transform where the Fourier coefficients $h_d(n)$ are the desired impulse response sequence of filter (IFS)

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega \longrightarrow (2)$$

The z transform of the sequence is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h_d(n)\, z^{-n} \longrightarrow (3)$$

(3) $\Rightarrow$ non causal digital filter of infinite duration

(iii) Truncate Impulse sequence to finite

To get an FIR filter transfer function, the series can be truncated by assigning

$$h(n) = h_d(n) \text{ for } |n| \le \frac{N-1}{2} \longrightarrow (4)$$

$$= 0, \quad \text{otherwise}$$

(iv) Take z transform to find H(z)

Then

$$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{N-1/2} h(n)\, z^{-n} \longrightarrow (5)$$

$$= h\left[\frac{N-1}{2}\right] z^{-(N-1)/2} + \dots + h(1)z^{-1} + h(0) + h(-1)z + h(-2)z^2 + \dots + h\left[-\left(\frac{N-1}{2}\right)\right] z^{(N-1)/2} \longrightarrow (6)$$

$$= h(0) + \sum_{n=1}^{N-1/2} \left[h(n)z^{-n} + h(-n)z^{n}\right] \longrightarrow (7)$$

For a symmetrical impulse response having symmetry at $n=0$

$$h(-n) = h(n) \longrightarrow (8)$$

Eqn (7) can be written as

$$\boxed{H(z) = h(0) + \sum_{n=1}^{N-1/2} h(n)\left[z^{n} + z^{-n}\right]} \longrightarrow (9)$$

The above transfer function is not physically realizable. Realizability can be brought by multiplying Eqn (9) by $z^{-(N-1)/2}$ where $\frac{N-1}{2}$ is delay in samples.

(V) Make $H(z)$ as a realizable filter

$$H'(z) = z^{-(N-1)/2}\, H(z)$$

$$H'(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=1}^{N-1/2} h(n)\left(z^{n} + z^{-n}\right)\right] \rightarrow$$

Problems on Fourier Series Method of Designing FIR filters :-

1) Design an ideal low pass filter with a freq response

$$H_d(e^{j\omega}) = 1 \quad \text{for } -\pi/2 \le \omega \le \pi/2$$

$$= 0 \quad \text{for } \pi/2 \le |\omega| < \pi$$

Find the values of $h(n)$ for $N=11$. Find $H(z)$. Plot

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the magnitude response

Soln

The freq response of low pass filter with $\omega_c = \pi/2$ is shown in Fig 1

Given step 1

$$H_d(e^{j\omega}) = 1 \text{ for } -\pi/2 \le \omega \le \pi/2$$
$$= 0 \text{ for } \pi/2 \le |\omega| \le \pi$$

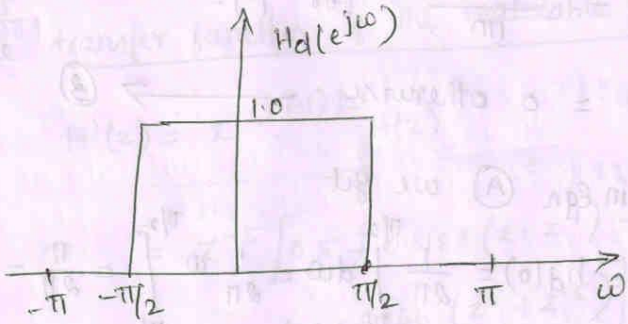

Fig 1: Ideal freq response of problem 1

∴ From freq response it is known that $\alpha = 0$.

we get a non causal filter coefficients symmetrical about $n=0$ (ie) $h_d(n) = h_d(-n)$

⇒ step 2

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega \longrightarrow ①$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\, d\omega \longrightarrow Ⓐ$$

$$= \frac{1}{2\pi j n} e^{j\omega} \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi n(2j)} \left[e^{j\pi n/2} - e^{-j\pi n/2} \right]$$

$$h_d(n) = \frac{\sin \pi/2\, n}{\pi n}, \quad -\infty \le n \le \infty \longrightarrow (2)$$

Step 3

Truncating $h_d(n)$ to 11 samples =)

$$h(n) = \frac{\sin \pi/2\, n}{\pi n} \quad \text{for } |n| \le 5$$

$$h(n) = 0 \text{ otherwise} \longrightarrow (3)$$

For $n=0$, in Eqn (A) we get

$$h(0) = h_d(0) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\omega = \frac{1}{2\pi} \omega \Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{2\pi} = 1/2$$

Step 4 : Filter Coefficients (Since symmetry)

For $n=1$

$$h(1) = h(-1) = \frac{\sin \pi/2}{\pi} = \frac{1}{\pi} = 0.3183$$

Ill^ly

$$h(2) = h(-2) = \frac{\sin \pi}{2\pi} = 0$$

$$h(3) = h(-3) = \frac{\sin 3\pi/2}{3\pi} = -\frac{1}{3\pi} = -0.106$$

$$h(4) = h(-4) = \frac{\sin 4\pi/2}{4\pi} = 0$$

$$h(5) = h(-5) = \frac{\sin 5\pi/2}{5\pi} = \frac{1}{5\pi} = 0.06366$$

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The transfer function of the filter is gn by

$$H(z) = h(0) + \sum_{n=1}^{N-1/2} \left[h(n)(z^n + z^{-n})\right]$$

$$= 0.5 + \sum_{n=1}^{5} h(n)\left[z^n + z^{-n}\right]$$

$$H(z) = 0.5 + 0.3183\left[z^{-1} + z^{1}\right] - 0.106\left[z^{-3} + z^{3}\right] + 0.06366\left[z^5 + z^{-5}\right]$$

Step 5
The transfer function of the realizable filter is

$$H'(z) = z^{-(N-1)/2} H(z)$$

$$= z^{-5}\left[0.5 + 0.3183(z + z^{-1}) - 0.106(z^3 + z^{-3}) + 0.06366(z^5 + z^{-5})\right]$$

$$H'(z) = 0.06366 + -0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6} - 0.106z^{-8} + 0.06366z^{-10} \quad \text{(4)}$$

From eqn (4), the filter coefficients are given by H(z)

$h(0) = h(10) = 0.06366 \iff \boxed{h(0) = h(11-1-0) = h(10)}$

$h(1) = h(9) = 0$

$h(2) = h(8) = -0.106$

$h(3) = h(7) = 0$

$h(4) = h(6) = 0.3183$

$h(5) = 0.5$

To find Magnitude response, the following eq have to be satisfied

Mag response

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n) \cos \omega n \quad \text{where}$$

$$a(0) = h\left[\frac{N-1}{2}\right] = h(5) = 0.5$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$$a(1) = 2h(5-1) = 2h(4) = 0.6366$$

$$a(2) = 2h(5-2) = 2h(3) = 0$$

$$a(3) = 2h(5-3) = 2h(2) = -0.212$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0.127$$

$$\bar{H}(e^{j\omega}) = 0.5 + 0.6366 \cos\omega + 0 - 0.212 \cos 3\omega + 0 + 0.127 \cos 5\omega$$

Magnitude Response in dB

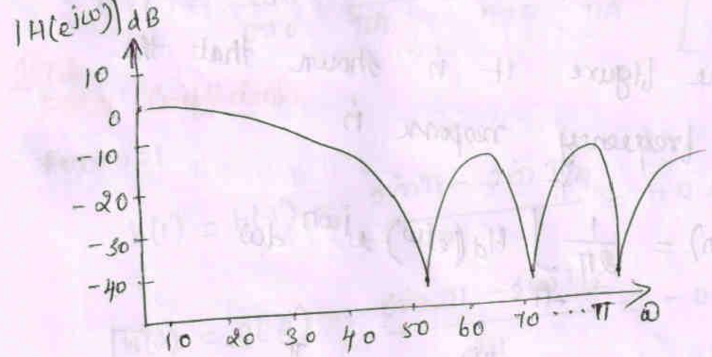
The Magnitude in dB is calculated by varying ω from 0 to π and is tabulated below

$$|\bar{H}(e^{j\omega})|_{dB} = 20 \log |\bar{H}(e^{j\omega})|$$

(17)

ω in degrees	0	20	180
$\lvert \bar{H}(e^{j\omega})\rvert$	1.057	0.970	-0.0516
$\lvert \bar{H}(e^{j\omega})\rvert_{dB}$	0.4	-0.263	-26

Magnitude Response Plot is shown below



2) Design an Ideal high pass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for } \pi/4 \le |\omega| \le \pi$$

$$= 0 \quad \text{for } |\omega| \le \pi/4$$

Find the values of $h(n)$ for $N=11$. Find $H(z)$. Plot the Magnitude response

Soln
The diagram is shown below which gives freq response of HPF

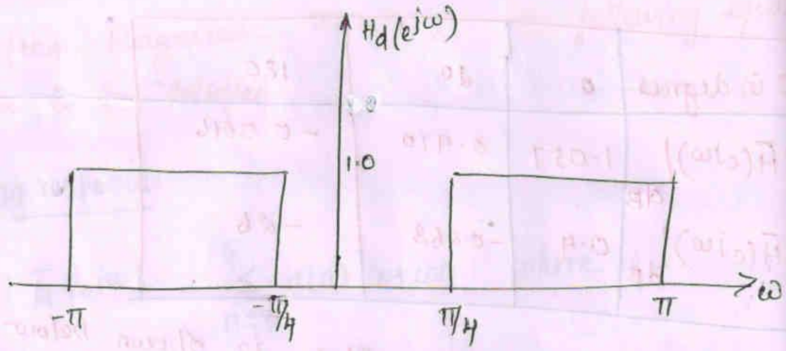


Fig1 ideal freq response of HPF

<u>step1</u>

From the figure it is shown that the desired frequency response is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/4} e^{j\omega n}\, d\omega + \int_{\pi/4}^{\pi} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi jn}\left\{\left[e^{j\omega n}\right]_{-\pi}^{-\pi/4} + \left[e^{j\omega n}\right]_{\pi/4}^{\pi}\right\}$$

$$= \frac{1}{(2j)\pi n}\left[e^{-j\pi n/4} - e^{-j\pi n} + e^{j\pi n} - e^{j\pi n/4}\right]$$

$$= \frac{1}{\pi n}\left[\sin \pi n - \sin \frac{\pi}{4} n\right], \quad -\infty < n < \infty \quad \text{①}$$

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Step 2

Truncating hd(n) to 11 samples, we have

$$h(n) = h_d(n) \text{ for } |n| \le 5$$
$$= 0, \text{ otherwise}$$

For n=0

$$h(0) = \lim_{n\to 0} \frac{\sin \pi n}{\pi n} - \lim_{n\to 0} \frac{\sin \frac{\pi}{4} n}{\pi n} \quad \left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1, \ \lim_{\theta\to 0} \frac{\sin n\theta}{\theta} = n \right]$$

Step 3

Filter Coefficients

For n=1

$$h(1) = h(-1) = \frac{\sin \pi - \sin \pi/4}{\pi} = -0.225$$

$$h(2) = h(-2) = \frac{\sin 2\pi - \sin \pi/2}{2\pi} = -0.159$$

$$h(3) = h(-3) = \frac{\sin 3\pi - \sin \frac{3\pi}{4}}{3\pi} = -0.075$$

$$h(4) = h(-4) = \frac{\sin 4\pi - \sin \pi}{4\pi} = 0$$

$$h(5) = h(-5) = \frac{\sin 5\pi - \sin \frac{5\pi}{4}}{5\pi} = 0.045$$

Step 4

The transfer function of the filter is gn by

$$H(z) = h(0) + \sum_{n=1}^{N-1/2} \left[h(n) \left(z^n + z^{-n} \right) \right]$$

$$= 0.75 + \sum_{n=1}^{5} \left[h(n) \left(z^n + z^{-n} \right) \right]$$

$$H(z) = 0.75 - 0.225 \left(z + z^{-1} \right) - 0.159 \left(z^2 + z^{-2} \right) - 0.075 \left(z^3 + z^{-3} \right) + 0.045 \left(z^5 + z^{-5} \right) \longrightarrow ②$$

Step 5

The transfer function of the realizable filter is

$$H'(z) = z^{-5} H(z)$$

$$= z^{-5}\left[0.75 - 0.225\left(z+z^{-1}\right) - 0.159\left(z^{2}+z^{-2}\right) - 0.075\left(z^{3}+z^{-3}\right) + 0.045\left(z^{5}+z^{-5}\right)\right]$$

$$H'(z) = 0.045 - 0.075z^{-2} - 0.159z^{-3} - 0.225z^{-4} + 0.75z^{-5} - 0.225z^{-6} - 0.159z^{-7} - 0.075z^{-8} + 0.045z^{-10} \longrightarrow (3)$$

Step 6

From eqn (3), the filter coefficients of causal filter by $H(z)$ is

Step 7 : Mag Response

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{N-1/2} a(n)\cos\omega n \quad \text{where}$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$$a(1) = 2h(5-1) = 2h(4) = -0.45$$

$$a(2) = 2h(5-2) = 2h(3) = -0.318$$

$$a(3) = 2h(5-3) = 2h(2) = -0.15$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0.09$$

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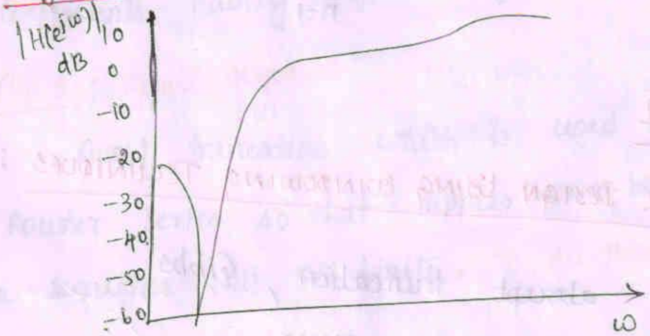
$$\bar{H}(e^{j\omega}) = a(0) + a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega + a(4)\cos 4\omega + a(5)\cos 5\omega$$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.45\cos\omega - 0.318\cos 2\omega - 0.15\cos 3\omega + 0.09\cos 5\omega \longrightarrow (4)$$

ω (in deg)	0	10	20	 180
$\bar{H}(e^{j\omega})$	-0.08	-0.66	-0.0086	. . . . . 0.94
$\lvert\bar{H}(e^{j\omega})\rvert$ dB	-22	-23.62	-41.3	. . . . . -0.537

Mag Response Plot



HW

3) Design an ideal band pass filter with a freq response

$$H_d(e^{j\omega}) = 1 \quad \text{for } \pi/4 \le |\omega| \le 3\pi/4$$
$$= 0, \quad \text{otherwise}$$

Find the values of $h(n)$ for $N = 11$ & plot freq response [Hint :- Freq Response $= e^{-j\omega(N-1)/2}\left[\sum_{n=0}^{N-1/2} a(n)\cos\omega n\right]$

$= \left[h(0) + \sum^{N-1/2} 2h(n)\cos\omega n\right]$]

4) Design an ideal bandreject filter with a desired freq response

$$H_d(e^{j\omega}) = 1 \text{ for } |\omega| \le \pi/3 \text{ \& } |\omega| \ge \frac{2\pi}{3}$$

$$= 0, \text{ otherwise}$$

Find the value of $h(n)$ for $N = 11$. Find $H(z)$. Plot the Magnitude Response

Hint :-

$$\left[\text{Freq response} = e^{-j\omega(N-1)/2}\left[\sum_{n=0}^{N-1/2} a(n)\cos\omega n\right.\right.$$

(or)

$$\left. h(0) + \sum_{n=1}^{N-1/2} 2h(n)\cos\omega n \right]$$

Topic III

III.1) FILTER DESIGN USING WINDOWING TECHNIQUES :

Due to abrupt truncation, Gibbs phenomenon (ie) ripples occur in Fourier series, To avoid this windowing technique is used.

III.2) PROCEDURE TO DESIGN FILTER USING WINDOWING TECHNIQUES

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Step 1 : Desired freq response

The desired frequency response $H_d(e^{j\omega})$ of a filter can be expanded in Fourier series. The resultant series is gn by

$$\boxed{H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)\, e^{-j\omega n}} \longrightarrow (1)$$

whereas Step 2: Infinite / Impulse Response sequence

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega \longrightarrow (2)$$

where $h_d(n)$ is also called as Fourier Coefficients having infinite length.

Step 3 : Finite sequence

Avoid truncation which is used in Filter design by Fourier series so that ripples can be avoided. The sequence will be finite. To do this windowing technique is used

$$\left.\begin{aligned} w(n) = w(-n) &\neq 0 \quad \text{for } |n| \leq \left(\frac{N-1}{2}\right) \\ &= 0 \quad \text{for } |n| > \left(\frac{N-1}{2}\right) \end{aligned}\right\} \longrightarrow (3)$$

Multiply window sequence $w(n)$ with $h_d(n)$ that we get a finite duration sequence $h(n)$ that satisfies the desired magnitude response

$$\boxed{h(n) = h_d(n)\, w(n)} \quad \text{for all } |n| \leq \left(\frac{N-1}{2}\right) \longrightarrow (4)$$

$$= 0 \quad \text{for } |n| > \left(\frac{N-1}{2}\right)$$

Step 4 : Transfer function of the filter

$$\boxed{H(z) = h(0) + \sum_{n=1}^{N-1/2} h(n)\left[z^{n} + z^{-n}\right]} \longrightarrow (5)$$

Step 5 : Transfer function of the realizable filter is

$$\boxed{H'(z) = z^{-(N-1)/2}\, H(z)}$$

Step 6

The frequency response $H(e^{j\omega})$ of the filter can be obtained by convolution of $H_d(e^{j\omega})$ and $W(e^{j\omega})$ given by

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W\left(e^{j(\omega-\theta)}\right) d\theta$$

$$\boxed{H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})}$$

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Diagramatic Representation of windowing Technique

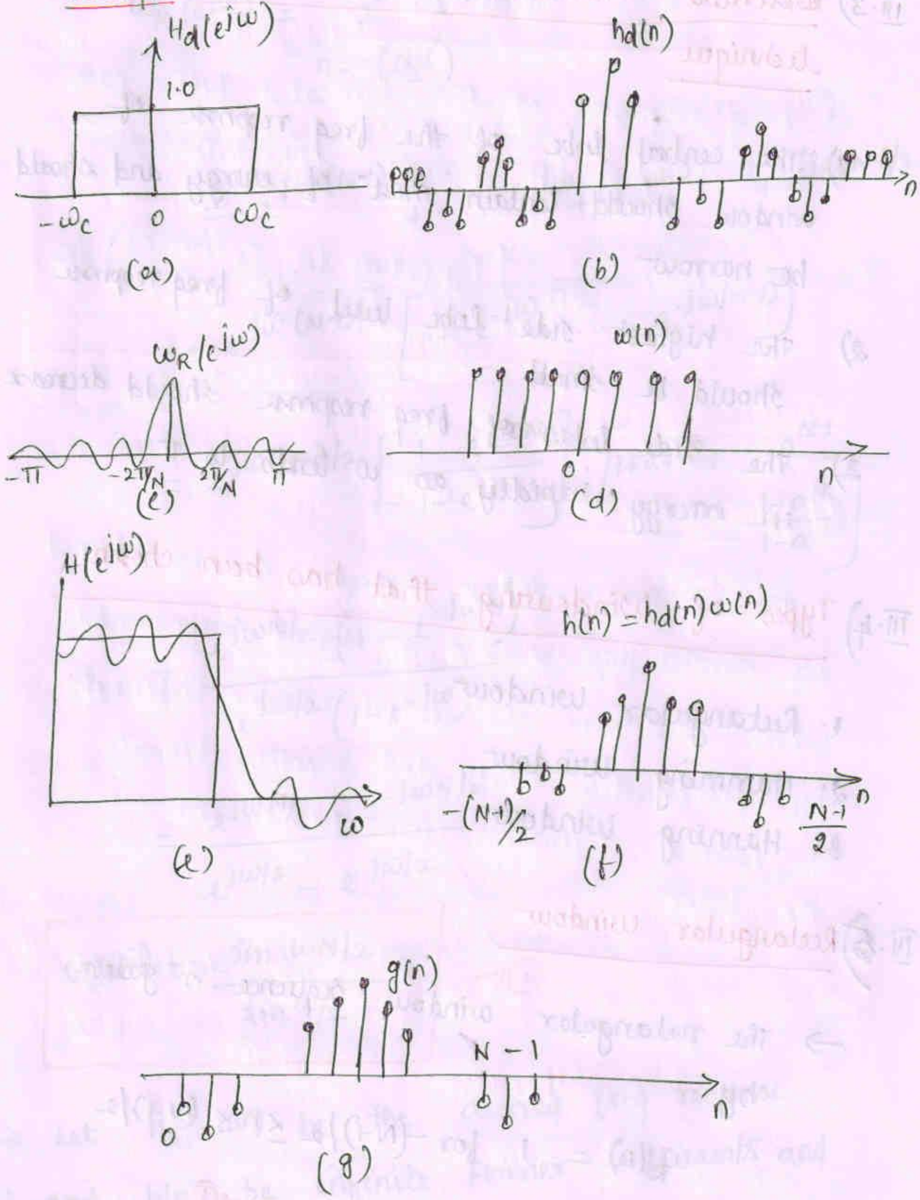

Fig 1: Windowing Technique

III.3) Desirable characteristics of windowing technique

1) The central lobe of the freq response of window should contain most of energy and should be narrow
2) The highest side lobe level of freq response should be small
3) The side lobes of freq response should decrease in energy rapidly as ω tends to π

III.4) Types of Windowing that has been chosen

1. Rectangular window
2. Hamming window
3. Hanning window

III.5) Rectangular window

→ The rectangular window sequence is given by

$$w_R(n) = 1 \quad \text{for } -(N-1)/2 \leq n \leq (N-1)/2$$
$$= 0 \quad \text{otherwise} \quad \rightarrow (1)$$

→ The spectrum of the rectangular window is given by

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$$W_R(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{N-1/2} e^{-j\omega n}$$

$$= e^{j\omega(N-1)/2} + \dots + e^{j\omega} + 1 + e^{-j\omega} + \dots + e^{-j\omega(N-1)/2}$$

$$= e^{j\omega(N-1)/2}\left[1 + e^{-j\omega} + \dots + e^{-j\omega(N-1)}\right]$$

$$= e^{j\omega(N-1)/2}\left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right] \quad \because \left[1+a+a^2+\dots a^{N-1} = \frac{1-a^N}{1-a}\right]$$

$$= \frac{e^{j\omega N/2}\left(1-e^{-j\omega N}\right)}{e^{j\omega/2}\left(1-e^{-j\omega}\right)}$$

$$= \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$\boxed{W_R(e^{j\omega}) = \frac{\sin \omega N/2}{\sin \omega/2}} \rightarrow (2)$$

→ Let $H_d(e^{j\omega})$ be the desired freq response and $h_d(n)$ be infinite Fourier Coefficients and to get finite impulse response filter, multiply $h_d(n)$ with rectangular window (ie)

$$\boxed{h(n) = h_d(n)\, w_R(n)} \rightarrow (3)$$

→ Freq response of the truncated filter can be obtained by periodic convolution

$$\boxed{H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, W_R(e^{j(\omega-\theta)})\, d\theta} \rightarrow (4)$$

III.6) Hanning Window

The Hanning window sequence can be obtained by substituting $\alpha = 0.5$ in Raised Cosine window which multiplies the central Fourier coefficients by approximately unity and smoothly truncate the Fourier coefficients towards the ends of the filter

$$w_\alpha(n) = \alpha + (1-\alpha)\cos\frac{2\pi n}{N-1} \quad \text{for } -(N-1)/2 \leq n \leq (N-1)/2 \rightarrow (1)$$

$$= 0 \text{ , otherwise}$$

sub $\alpha = 0.5$ in (1)

$$\boxed{\begin{aligned} w_{H_n}(n) &= 0.5 + (1-0.5)\cos\frac{2\pi n}{N-1} \quad \text{for } -(N-1)/2 \leq n \leq (N-1)/2 \\ &= 0 \quad \text{otherwise} \end{aligned}}$$

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The frequency response of Hanning window is given by following eqn by sub $\alpha = 0.5$

$$W_\alpha(e^{j\omega}) = \alpha \frac{\sin \omega N/2}{\sin \omega/2} + \frac{1-\alpha}{2} \frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))}$$

$$+ \frac{1-\alpha}{2} \frac{\sin(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi(N-1))} \longrightarrow (2)$$

$\therefore \alpha = 0.5$ for (2) $\Rightarrow$

$$\boxed{W_{Hn}(e^{j\omega}) = 0.5 \frac{\sin \frac{\omega N}{2}}{\sin \omega/2} + 0.25 \frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))} + 0.25 \frac{\sin(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))}}$$

III-ii) Hamming Window

The Hamming window sequence can be obtained by substituting $\alpha = 0.54$ in (1) below

$$w_\alpha(n) = \alpha + (1-\alpha)\cos\frac{2\pi n}{N-1} \quad \text{for } -(N-1)/2 \le n \le \frac{(N-1)}{2}$$

$$= 0 \quad \text{otherwise} \longrightarrow (1)$$

$$\boxed{\begin{aligned} w_H(n) &= 0.54 + 0.46\cos(2\pi n/N-1) \quad \text{for } -(N-1)/2 \le n \le \frac{(N-1)}{2} \\ &= 0 \quad \text{otherwise} \end{aligned}}$$

The freq response of Hamming window can be obtained from

$$W_\alpha(e^{j\omega}) = \alpha \frac{\sin \omega N/2}{\sin \omega/2} + \frac{1-\alpha}{2} \frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))}$$

$$+ \frac{1-\alpha}{2} \frac{\sin(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))} \rightarrow (2)$$

Sub $\alpha = 0.54$ in (2)

$$\boxed{W_H(e^{j\omega}) = 0.54 \frac{\sin \omega N/2}{\sin \omega/2} + 0.23 \frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))} + 0.23 \frac{\sin(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))}}$$

Problems on windowing Techniques

1) Design an ideal high pass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for } \pi/4 \le |\omega| \le \pi$$
$$= 0 \quad \text{for } |\omega| \le \pi/4$$

Find the values of $h(n)$ for $N=11$. Find $H(z)$. Plot the mag response. Implement it using a) Hamming window b) Hanning window

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Soln

Gn

Step 1

$$H_d(e^{j\omega}) = 1 \quad \text{for } \pi/4 \le |\omega| \le \pi$$

$$= 0 \quad \text{for } |\omega| \le \pi/4$$

Step 2

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi j n}\left\{\left[e^{j\omega n}\right]_{-\pi}^{-\pi/4} + \left[e^{j\omega n}\right]_{\pi/4}^{\pi}\right\}$$

$$= \frac{1}{(2j)\pi n}\left[e^{-j\pi n/4} - e^{-j\pi n} + e^{j\pi n} - e^{j\pi n/4}\right]$$

$$h_d(n) = \frac{1}{\pi n}\left[\sin \pi n - \sin \pi/4\, n\right], \quad -\infty \le n \le \infty \quad \to (1)$$

Step 3

a) Hanning window

$$\omega_{Hn}(n) = 0.5 + 0.5 \cos \frac{2\pi n}{N-1} \quad \text{for } -(N-1)/2 \le n \le (N-1)/2$$

$$= 0, \quad \text{otherwise}$$

For $N=11$

$$w_{Hn}(n) = 0.5 + 0.5\cos\frac{\pi n}{5}, \quad -5 \le n \le 5$$
$$= 0, \text{ otherwise}$$

$$w_{Hn}(0) = 0.5 + 0.5 = 1$$

$$w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5\cos\frac{\pi}{5} = 0.9045$$

$$w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5\cos\frac{2\pi}{5} = 0.655$$

$$w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5\cos\frac{3\pi}{5} = 0.345$$

$$w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5\cos\frac{4\pi}{5} = 0.0945$$

$$w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5\cos\pi = 0$$

Step 4

The filter coefficients can be obtained by

$$h_d(n) = \frac{\sin\pi n - \sin\frac{\pi}{4}n}{\pi n}$$

$$h_d(0) = \lim_{n\to 0}\left[\frac{\sin\pi n - \sin\frac{\pi}{4}n}{\pi n}\right] = 1 - \frac{1}{4} = 0.75$$

$$h_d(-1) = h_d(1) = \frac{\sin\pi - \sin\pi/4}{\pi} = -0.225$$

$$h_d(-2) = h_d(2) = \frac{\sin 2\pi - \sin\pi/2}{2\pi} = -0.159$$

$$h_d(-3) = h_d(3) = \frac{\sin 3\pi - \sin 3\pi/4}{3\pi} = -0.075$$

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$$h_d(-4) = h_d(4) = \frac{\sin 4\pi - \sin \pi}{4\pi} = 0$$

$$h_d(-5) = h_d(5) = \frac{\sin 5\pi - \sin 5\pi/4}{5\pi} = 0.045$$

Step 5

The filter coefficients using Hanning window are (ie) to make finite

$$h(n) = h_d(n)\, w_{Hn}(n) \quad \text{for } -5 \le n \le 5$$
$$= 0 \quad \text{otherwise}$$

$$h(0) = h_d(0)\, w_{Hn}(0) = (0.75)(1) = 0.75$$

$$h(-1) = h(1) = h_d(1)\, w_{Hn}(1) = (-0.225)(0.905) = -0.204$$

$$h(-2) = h(2) = h_d(2)\, w_{Hn}(2) = (-0.159)(0.655) = -0.104$$

$$h(-3) = h(3) = h_d(3)\, w_{Hn}(3) = (-0.075)(0.345) = -0.026$$

$$h(-4) = h(4) = h_d(4)\, w_{Hn}(4) = (0)(0.8145) = 0$$

$$h(-5) = h(5) = h_d(5)\, w_{Hn}(5) = (0.045)(0) = 0$$

Step 6

The transfer function of the filter is gn by

$$H(z) = h(0) + \sum_{n=1}^{5} h(n)\left[z^{n} + z^{-n}\right]$$

$$H(z) = 0.75 - 0.204\,(z + z^{-1}) - 0.104\,(z^{2} + z^{-2}) - 0.026\,(z^{3} + z^{-3})$$

Step 7

The transfer function of the realizable filter is

$$H'(z) = z^{-5} H(z)$$

$$= -0.026 z^{-2} - 0.104 z^{-3} - 0.204 z^{-4} + 0.75 z^{-5} - 0.204 z^{-6} - 0.104 z^{-7} - 0.026 z^{-8}$$

Step 8

The causal filter coefficients are

$h(0) = h(1) = h(9) = h(10) = 0$

$h(2) = h(8) = -0.026$

$h(3) = h(7) = -0.104$

$h(4) = h(6) = -0.204$

$h(5) = 0.75$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{N-1/2} a(n) \cos \omega n$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$a(1) = 2h(5-1) = 2h(4) = -0.408$

$a(2) = 2h(5-2) = 2h(3) = -0.208$

$a(3) = 2h(5-3) = 2h(2) = -0.052$

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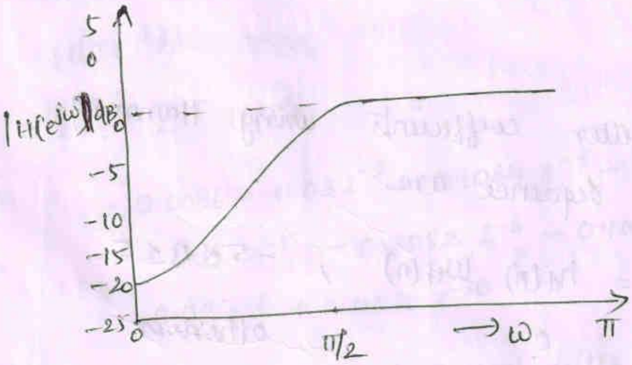
$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\therefore \bar{H}(e^{j\omega}) = 0.75 - 0.408\cos\omega - 0.208\cos 2\omega - 0.052\cos 3\omega$$

ω in degrees	0	15	30	45	60	75	90	180
$\bar{H}(e^{j\omega})$	0.082	0.139	0.292	0.498	0.702	0.86	0.96	1.002
$\lvert H(e^{j\omega})\rvert$ in dB	-21.72	-17.14	-10.67	-6.05	-3.07	-1.297	-0.3726	0.017

Mag Response using Hanning window



b) Hamming window

Step 1 The Hamming window sequence is given by

$$w_H(n) = 0.54 + 0.46\cos\frac{2\pi n}{N-1} \quad \text{for } -(N-1)/2 \le n \le \frac{(N-1)}{2}$$

$$= 0 \quad \text{otherwise}$$

The window sequence for $N=11$ is gn by

$$w_H(n) = 0.54 + 0.46 \cos \frac{\pi n}{5} \quad \text{for } -5 \le n \le 5$$
$$= 0 \quad , \text{ otherwise}$$

$w_H(0) = 0.54 + 0.46 = 1$

$w_H(-1) = w_H(1) = 0.54 + 0.46 \cos \pi/5 = 0.912$

$w_H(-2) = w_H(2) = 0.54 + 0.46 \cos \frac{2\pi}{5} = 0.682$

$w_H(-3) = w_H(3) = 0.54 + 0.46 \cos \frac{3\pi}{5} = 0.398$

$w_H(-4) = w_H(4) = 0.54 + 0.46 \cos \frac{4\pi}{5} = 0.1678$

$w_H(-5) = w_H(5) = 0.54 + 0.46 \cos \pi = 0.08$

Step 2

The filter coefficients using Hamming window sequence are

$$h(n) = h_d(n)\, w_H(n) \quad , \quad -5 \le n \le 5$$
$$0 \qquad \text{otherwise}$$

$h(0) = h_d(0)\, w_{Hn}(0) = (1)(0.75) = 0.75$

$h(-1) = h(1) = h_d(1)\, w_{Hn}(1) = (-0.225)(0.912) = -0$

$h(-2) = h(2) = h_d(2)\, w_{Hn}(2) = (-0.159)(0.682) = -0$

$h(-3) = h(3) = h_d(3)\, w_{Hn}(3) = (-0.075)(0.398) = -0$

(37)

$h(-4) = h(4) = h_d(4)\ w_{Hn}(4) = (0)(0.1678) = 0$

$h(-5) = h(5) = h_d(5)\ w_{Hn}(5) = (-0.045)(0.08) = 0.0036$.

Step 3

The transfer function of filter is gn by

$$H(z) = h(0) + \sum_{n=1}^{5} [h(n)(z^{-n} + z^{n})]$$

$$H(z) = 0.75 - 0.2052(z^{-1}+z) - 0.1084(z^{-2}+z^{2}) - 0.03(z^{-3}+z^{3}) + 0.0036(z^{-5}+z^{5})$$

Step 4

The transfer function of the realizable filter is

$$H'(z) = z^{-5} H(z)$$

$$= 0.0036 - 0.03z^{-2} - 0.1084z^{-3} - 0.2052z^{-4} + 0.75z^{-5} - 0.2052z^{-6} - 0.1084z^{-7} - 0.03z^{-8} + 0.0036z^{-10}$$

The filter coefficients of causal filter are

$h(0) = h(10) = 0.0036$, $h(1) = h(9) = 0$, $h(2) = h(8) = -0.03$

$h(3) = h(7) = -0.1084$, $h(4) = h(6) = -0.2052$;

$h(5) = 0.75$

Step 5: Magnitude Response

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{N-1/2} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$$a(1) = 2h(5-1) = 2h(4) = -0.4104$$

$$a(2) = 2h(5-2) = 2h(3) = -0.2168$$

$$a(3) = 2h(5-3) = 2h(2) = -0.06$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0.0072$$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.4104\cos\omega - 0.2168\cos 2\omega - 0.06\cos 3\omega + 0.0072\cos 5\omega$$

ω in degrees	0	15	30	45	60	[illegible]
$\bar{H}(e^{j\omega})$	0.07	0.125	0.28	0.497	0.7168	[illegible]
$\lvert H(e^{j\omega})\rvert$ dB	-23.1	-18	-11	-6.07	-2.89	

Step 6: Magnitude Response

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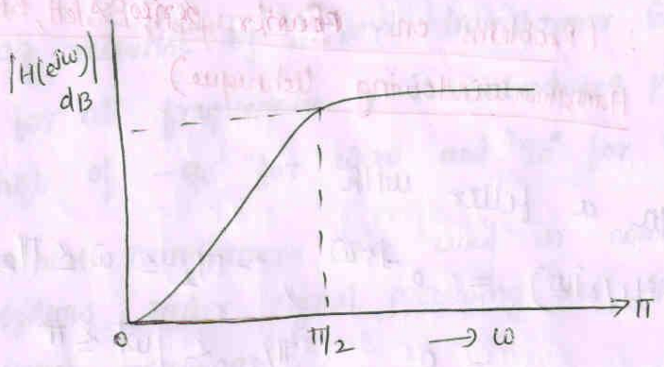

III.8) Other Windowing Technique

1) Triangular or Barlett Window

N point triangular window is given by

$$W_T(n) = 1 - \frac{2|n|}{N-1} \quad \text{for } -(N-1)/2 \le n \le (N-1)/2$$

Frequency response is gn by

$$W_T(e^{j\omega}) = \left[\frac{\sin\left(\frac{N-1}{4}\right)\omega}{\sin \omega/2} \right]^2$$

2) Blackmann window

The Blackman window sequence is given by

$$\omega_B(n) = 0.42 + 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1}; \quad -(N-1)/2 \le n \le (N-1)/2$$

$$= 0, \text{ otherwise}$$

HW : (Problems on Rectangular, Barlett, Blackman, Hann, Hamm windowing technique)

2) Design a filter with

$$H_d(e^{j\omega}) = e^{-j5\omega}, \quad -\pi/2 \le \omega \le \pi/2$$
$$= 0, \quad \pi/2 < |\omega| \le \pi$$

using Blackmann window with $N = 11$

3) For the desired response

$$H_d(e^{j\omega}) = e^{-j3\omega}, \quad -\pi/8 \le \omega \le \pi/8$$
$$= 0, \quad \pi/8 \le |\omega| \le \pi$$

Determine $H(e^{j\omega})$ for $N = 7$ and compare the responses for (a) Rectangular window b) Hanning window c) Hamming window

III.9) Hilbert Transformers

The frequency response of an ideal transformer is given by

$$H_d(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi \\ j, & -\pi < \omega < 0 \end{cases}$$

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Mag response of Hilbert transformer is 1 for all frequencies, it introduces phase shift of -90° for $\omega > 0$ and 90° for $\omega < 0$

→ Hilbert Transformers are used in communication systems, radar signal processing and speech signal processing

The impulse response of an ideal Hilbert Transformer is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} j\, e^{j\omega n}\, d\omega - \int_{0}^{\pi} j\, e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[j\,\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{0}\left[-j\,\frac{e^{j\omega n}}{jn}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi n}\left[1 - e^{-j\pi n} - e^{j\pi n} + 1\right]$$

$$= \frac{1}{2\pi n}\left[2 - \left(e^{j\pi n} + e^{-j\pi n}\right)\right]$$

$$h_d(n) = \frac{1}{2\pi n}\left[2 - 2\cos\pi n\right] = \frac{1-\cos\pi n}{\pi n}$$

$$= \frac{2\sin^2(\pi n/2)}{\pi n} \quad \text{for } n \neq 0$$

$$= 0 \quad \text{for } n = 0$$

$$h_d(n) = \begin{cases} \frac{2\sin^2(\pi n/2)}{\pi n}, & n \neq 0 \\ 0 & n = 0 \end{cases}$$

$h_d(n)$ is infinite in duration and non causal

The frequency response a linear phase Hilbert transformer is given by

$$H_d(e^{j\omega}) = -je^{-j\alpha\omega}, \quad 0 < \omega < \pi$$

$$= je^{-j\alpha\omega}, \quad -\pi < \omega < 0 \quad \longrightarrow (1)$$

where $\alpha = \frac{N-1}{2}$

The filter coefficients are antisymmetrical about $n=0$, satisfying $h_d(n) = -h_d(-n)$ always for Hilbert transformer. This can be given from eqn (1) and is given in Fig

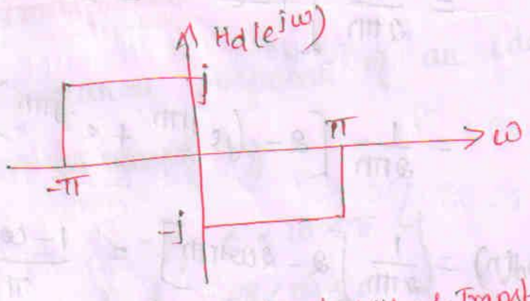

Fig 1: Antisymmetrical Hilbert Transformer

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Problems on Hilbert Transformer & windowing Technique :-

1) Design an ideal Hilbert transformer having frequency response

$$H(e^{j\omega}) = j \quad \text{for } -\pi \leq \omega \leq 0$$
$$= -j \quad \text{for } 0 \leq \omega < \pi$$

Using a) rectangular window b) Blackman window For N=11 plot the freq response in both cases

Soln

The ideal frequency response is given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega \longrightarrow ①$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} j e^{j\omega n} d\omega + \int_{0}^{\pi} -j e^{j\omega n} d\omega\right.$$

$$h_d(n) = \frac{1-\cos\pi n}{\pi n}$$

The filter coefficients are antisymmetrical about n = 0, satisfying $h_d(n) = -h_d(-n)$

For $n=0$

$$h_d(0) = \frac{j}{2\pi}\left[\int_{-\pi}^{0} d\omega + \int_{0}^{\pi} d\omega\right]$$

$$= \frac{j}{2\pi}\left[0+\pi-\pi-0\right] = 0$$

$$h_d(1) = -h_d(-1) = \frac{1-\cos\pi}{\pi} = \frac{2}{\pi}$$

$$h_d(2) = -h_d(-2) = \frac{1-\cos 2\pi}{2\pi} = 0$$

$$h_d(3) = -h_d(-3) = \frac{1-\cos 3\pi}{3\pi} = \frac{2}{3\pi}$$

$$h_d(4) = -h_d(-4) = \frac{1-\cos 4\pi}{4\pi} = 0$$

$$h_d(5) = -h_d(-5) = \frac{1-\cos 5\pi}{5\pi} = \frac{2}{5\pi}$$

a) Rectangular window :-

i) $h(n) = h_d(n)\, w_R(n) = h_d(n)$ for $-5 \le n \le 5$

$$h(1) = -h(-1) = 2/\pi$$

$$h(2) = -h(-2) = 0$$

$$h(3) = -h(-3) = 2/3\pi$$

$$h(4) = -h(-4) = 0$$

$$h(5) = -h(-5) = 2/5\pi$$

(45)

2) The transfer function of the Hilbert transformer is

$$H(z) = \frac{2}{\pi}(z - z^{-1}) + \frac{2}{3\pi}(z^{3} - z^{-3}) + \frac{2}{5\pi}(z^{5} - z^{-5})$$

3) The realizable transfer function

$$H'(z) = z^{-5} H(z)$$

$$= \frac{2}{5\pi} + \frac{2}{3\pi} z^{-2} + \frac{2}{\pi} z^{-4} - \frac{2}{\pi} z^{-6} - \frac{2}{3\pi} z^{-8} - \frac{2}{5\pi} z^{-10}$$

$$= 0.127 + 0.2122 z^{-2} + 0.6366 z^{-4} - 0.6366 z^{-6} - 0.2122 z^{-8} - 0.127 z^{-10}$$

4) Causal sequence of Hilbert transformer is

$$h(0) = -h(10) = 0.1273$$

$$h(1) = h(9) = h(3) = h(7) = h(5) = 0$$

$$h(2) = -h(8) = 0.2122$$

$$h(4) = -h(6) = 0.6366$$

5) $H(e^{j\omega}) = \sum_{n=1}^{5} c(n) \sin \omega n$

$$c(n) = 2h\left(\frac{N-1}{2} - n\right)$$

$$c(1) = 2h(4) = 1.2732$$

$$c(2) = 2h(3) = 0$$

$$c(3) = 2h(2) = 0.4244$$

$c(4) = 2h(1) = 0$

$c(5) = 2h(0) = 0.2546$

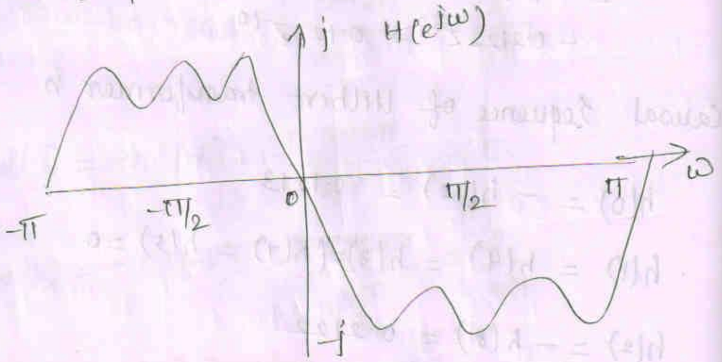
$\bar{H}(e^{j\omega}) = 1.2732 \sin\omega + 0.4244 \sin 3\omega + 0.2546 \sin$

$H(e^{j\omega}) = -j\bar{H}(e^{j\omega})$

$$\boxed{H(e^{j\omega}) = -j\,[1.2732 \sin\omega + 0.4244 \sin 3\omega + 0.2546}$$

ω	0	10	20	30	45	..
$H(e^{j\omega})$	0	$-j0.627$	$-j1.05$	$-j1.19$	$-j1.02$	

∴ The frequency response is plotted as



b) Blackmann window

1) The Blackmann window sequence for $N = 11$ is

$$\omega_B(n) = 0.42 + 0.5 \cos\frac{\pi n}{5} + 0.08 \cos\frac{2\pi n}{5}$$

$$= 0, \text{ otherwise}$$

$w_B(0) = 1$

$w_B(1) = w_B(-1) = 0.849$

$w_B(2) = w_B(-2) = 0.509$

$w_B(3) = w_B(-3) = 0.2$

$w_B(4) = w_B(-4) = 0.04$

$w_B(5) = w_B(-5) = 0$

2) The coefficients of Hilbert transformer are

$h(n) = h_d(n)\, w_B(n)$ for $-5 \le n \le 5$

$= 0$, otherwise

$h(0) = h_d(0)\, w_B(0) = (0)(1) = 0$

$h(1) = -h(-1) = h_d(1)\, w_B(1) = \frac{2}{\pi}(0.849) = 0.5405$

$h(2) = -h(-2) = h_d(2)\, w_B(2) = (0)(0.509) = 0$

$h(3) = -h(-3) = h_d(3)\, w_B(3) = \left(\frac{2}{3\pi}\right)(0.2) = 0.0423$

$h(4) = -h(-4) = h_d(4)\, w_B(4) = (0)(0.04) = 0$

$h(5) = -h(-5) = h_d(5)\, w_B(5) = \left(\frac{2}{5\pi}\right)(0) = 0$

3) The realizable transformer is

$H'(z) = z^{-5} H(z)$

$= z^{-5}\left[0.54\,(z - z^{-1})\right] + 0.424\left[z^{3} - z^{-3}\right]$

$H'(z) = 0.0424\, z^{-2} + 0.54\, z^{-4} - 0.54\, z^{-6} - 0.042\, z^{-8}$

4) The causal coefficients of Hilbert transformer are

$h(0) = h(10) = h(1) = h(9) = h(3) = h(7) = h(5) = 0$

$h(2) = -h(8) = 0.424$

$h(4) = -h(6) = 0.54$

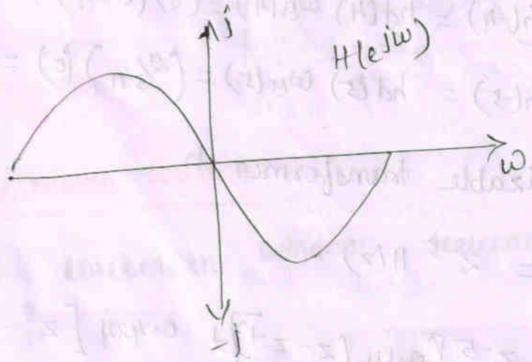
$$\bar{H}(e^{j\omega}) = \sum_{n=1}^{5} c(n) \sin n\omega$$

$$= 1.083 \sin\omega + 0.089 \sin 3\omega$$

$$H(e^{j\omega}) = -j\bar{H}(e^{j\omega})$$

5) Freq response plot

ω	0	15	30	180
$H(e^{j\omega})$	0	$-j0.339$	$-j0.624$	0



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Topic IV

Frequency Sampling Method for Designing FIR Filters :-

1) FILTER COEFFICIENTS

Let $h(n)$ is the filter coefficients of an FIR filter and $H(k)$ is DFT of $h(n)$

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{j2\pi kn/N} \quad n=0,1,\ldots N-1 \quad \rightarrow (1)$$

and

$$H(k) = \sum_{k=0}^{N-1} h(n)\, e^{-j2\pi kn/N}, \quad k=0,1,\ldots N-1 \quad \rightarrow (2)$$

2) Z Transform

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} \longrightarrow (3)$$

Sub (1) in (3)

$$H(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{j2\pi kn/N}\right] z^{-n}$$

$$= \sum_{k=0}^{N-1}\frac{H(k)}{N}\sum_{n=0}^{N-1} H(k)\left(e^{j2\pi k/N} z^{-1}\right)^{n}$$

$$= \sum_{k=0}^{N-1}\frac{H(k)}{N}\,\frac{1-\left(e^{j2\pi k/N} z^{-1}\right)^{N}}{1-e^{j2\pi k/N} z^{-1}}$$

$$H(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1-e^{j2\pi k/N} z^{-1}} \longrightarrow (4)$$

~~The freq response~~

3) Freq Response

The freq response of the FIR filter can be obtained by setting $z = e^{j\omega}$ in (4)

$$H(e^{j\omega}) = \frac{1-e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1-e^{j2\pi k/N} e^{-j\omega}}$$

$$= \frac{e^{-j\omega N/2}\left(e^{j\omega N/2} - e^{-j\omega N/2}\right)}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1-e^{-j(\omega - \ldots}}$$

$$= \frac{e^{-j\omega N/2}}{N} \sum_{k=0}^{N-1} \frac{H(k)\left(e^{j\omega N/2} - e^{-j\omega N/2}\right)}{e^{-j(\omega/2 - \pi k/N)}\left[e^{j(\omega/2 - \pi k/N)} - e^{-j(\omega/2 \ldots}\right.}$$

$$= \frac{e^{-j\omega(N-1)/2}}{N} \sum_{k=0}^{N-1} \frac{H(k) e^{-j\pi k/N} \sin \omega N/2}{\sin(\omega/2 - \pi k/N)} \longrightarrow$$

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$$= \frac{e^{-j\omega(N-1)/2}}{N} \sum_{k=0}^{N-1} H(k)\,(-1)^k\, e^{-j\pi k/N} \frac{\sin N(\omega/2 - k\pi/N)}{\sin(\omega/2 - \pi k/N)} \longrightarrow (6)$$

$$\left[\because \sin\left(\frac{\omega N}{2} - k\pi\right) = (-1)^k \sin\frac{\omega N}{2} \right]$$

IV.1) Design : Procedure

Two types of design

IV.1.1) Type I design :-

Here, the frequency samples of the desired response $H_d(e^{j\omega})$ are determined using the relation

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}, \quad k = 0, 1, \ldots N-1 \longrightarrow (7)$$

The frequency samples can be expressed in the form

$$\boxed{H(k) = |H(k)|\, e^{j\theta(k)}} \longrightarrow (8)$$

For linear phase

$$\theta(k) = -\alpha\omega \Big|_{\omega = \frac{2\pi k}{N}}$$

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k, \quad k = 0, 1, \ldots N-1 \longrightarrow (9)$$

Filter coefficients h(n) can be obtained by

$$\boxed{h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{j2\pi nk/N}}, \quad n=0,1\ldots N-1 \rightarrow (10)$$

H(k) is symmetry for N odd or even (real)

$$H(N-k) = H^*k, \quad k=0,1\ldots N-1 \rightarrow (A)$$

Important design (Type 2) formulas

1) Phase of Type 2 design ($\theta(k)$)

N = odd

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k, \quad k=0,1,2\ldots\frac{N-1}{2}$$

$$= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k, \quad k=\frac{N+1}{2}\ldots N-1 \rightarrow (11)$$

N = even

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k, \quad k=0,1,2\ldots N/2-1$$

$$= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k, \quad k=N/2+1\ldots N-1 \rightarrow (12)$$

$$= 0, \quad \text{otherwise}$$

2) The desired frequency response after consolidation can be gn by ($H_d(e^{j\omega})$)

(53)

N odd sub (11) in (8)

$$H(k) = |H(k)|\, e^{-j(N-1)\pi k/N}, \quad k = 0, 1 \ldots \frac{N-1}{2}$$

$$= |H(k)|\, e^{j[(N-1)\pi - (N-1)\pi k/N]}, \quad k = \frac{N+1}{2} \ldots N-1 \longrightarrow (13)$$

N even, sub (12) in (8)

$$H(k) = |H(k)|\, e^{-j(N-1)\pi k/N} \quad k = 0, 1 \ldots N/2 - 1$$

$$= |H(k)|\, e^{j[(N-1)\pi - (N-1)\pi k/N]}, \quad k = N/2 + 1 \ldots N-1$$

$$= 0, \quad k = N/2 \longrightarrow (14)$$

3) The filter coefficients that satisfies the symmetry condition is $h(n) = h(N-1-n)$ → (15)

$H(k)$ can be used to reduce the frequency specifications from N points to $\frac{N+1}{2}$ points for N odd and $N/2$ points for N even

By substituting (10) and (A), $h(n)$ for odd and even is given as

N odd

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{(N-1)/2} \mathrm{Re}\left[H(k)\, e^{j2\pi kn/N}\right]\right\} \rightarrow (16)$$

N even

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{N/2-1} \mathrm{Re}\left[H(k)\, e^{j2\pi kn/N}\right]\right\} \rightarrow (17)$$

∴ The system function of the filter can be obtained by

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} \longrightarrow (18)$$

IV.1.2) Type 2 Design

In this type of design the frequency samples H(K) are obtained using the relation

1) $$H(K) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}(k+1/2)}$$

$$H(K) = H_d\left(e^{j\pi(2k+1)/N}\right), \quad k = 0, 1 \ldots N-1 \rightarrow (19)$$

The initial point is located at $\omega = \pi/N$ and space b/w two pts is $\frac{2\pi}{N}$

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2) The filter coefficients can be obtained using the relation

$$h(n) = \sum_{k=0}^{N-1} H(k)\, e^{j2\pi kn/N}, \quad k = 0, 1 \ldots N-1$$

The condition that h(n) be real is

for N odd

$$H(N-k-1) = H^*(k), \quad k = 0, 1 \ldots \frac{N-1}{2} - 1$$

$$H\left(\frac{N-1}{2}\right) = 0 \longrightarrow (20)$$

N even

$$H(N-k-1) = H^*(k), \quad k = 0, 1 \ldots N/2 - 1 \rightarrow (21)$$

3) Then the filter coefficients for N, odd and even is

N odd

$$h(n) = \frac{2}{N} \sum_{k=0}^{N-3/2} \text{Re}\left[H(k)\, e^{jn\pi(2k+1)/N}\right] \rightarrow (22)$$

N even

$$h(n) = \frac{2}{N} \sum_{k=0}^{N/2-1} \text{Re}\left[H(k)\, e^{jn\pi(2k+1)/N}\right] \rightarrow (23)$$

Problems on Frequency Sampling Method

1) Determine the filter coefficients h(n) obtained by sampling

$$H_d(e^{j\omega}) = e^{-j(N-1)\omega/2}, \quad 0 \le |\omega| \le \pi/2$$
$$= 0, \quad \pi/2 \le |\omega| \le \pi$$

for N=7

Soln

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{7}}, \; k = 0, 1, 2 \ldots 6$$

$$H(0) = {}_d\,(e^{-j6\pi k/7})$$

$$= e^{-j6\pi \times 0/7} = 1$$

$$H(1) = e^{-j6\pi \times 1/7} = 1 \quad (j = -1)$$

$$H(2) = e^{-j6\pi \times 2/7} = 0$$

$$H(3) = e^{-j6\pi \times 3/7} = 0$$

$$H(4) = 0$$

$$H(5) = 0$$

$$H(6) = e^{-j6\pi \times 6/7} = 1$$

* It is also can be shown by figure (ideal mag respon

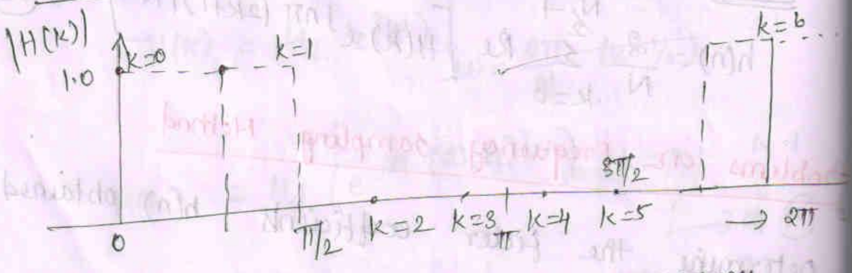

Fig1: Ideal Mag & phase response

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1) $|H(k)| = 1$ for $k = 0, 1, 6$

$= 0$ for $k = 2, 3, 4, 5$ $\longrightarrow$ ①

2) phase can be given by

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k = \frac{-6}{7}\pi k \quad \text{for } k = 0, 1, 2, 3$$

$$= +(N-1)\pi - \left(\frac{N-1}{N}\right)\pi k$$

$$= 6\pi - \frac{6\pi k}{7} = \frac{6\pi}{7}(7-k) \quad \text{for } k = 4, 5, 6.$$

$\hookrightarrow$ ②

$\cancel{= (N-1)\pi}$

3) Freq response can be obtained by substituting ① & ② in ⑧ (or) 13

$$\Rightarrow H(k) = e^{-j6\pi k/7}, \quad k = 0, 1$$

$$= 0, \quad k = 2, 3, 4, 5$$

$$= e^{-j6\pi(k-7)/7}, \quad k = 6$$

4) The filter coefficients for N odd are gn by

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{N-1/2} \mathrm{Re}\left[H(k)\, e^{j2\pi kn/7}\right]\right\}, \quad n = 0, 1, \ldots N-1$$

$$= \frac{1}{7}\left\{1 + 2\,\mathrm{Re}\left(e^{-j6\pi k/7}\, e^{j2\pi kn/7}\right)\right\}, \quad H(0) = 1$$

$$= \frac{1}{7}\left\{1 + 2\,Re\left(e^{j2\pi(n-3)/7}\right)\right\} \quad \left[e^{j\theta} = \cos\theta + j\sin\theta \quad \therefore Re[e^{j\theta}] = \cos\theta\right]$$

$$= \frac{1}{7}\left\{1 + 2\cos\frac{2\pi}{7}(n-3)\right\}$$

$$h(0) = h(6) = \frac{1}{7}\left\{1 + 2\cos\frac{6\pi}{7}\right\} = -0.11456$$

$$h(1) = h(5) = \frac{1}{7}\left\{1 + 2\cos\frac{4\pi}{7}\right\} = 0.07928$$

$$h(2) = h(4) = \frac{1}{7}\left\{1 + 2\cos\frac{2\pi}{7}\right\} = 0.321$$

$$h(3) = \frac{1}{7}(1+2) = 0.42857.$$

2) Using frequency sampling method, design a bandpass filter with the following specifications

Soln

Sampling frequency $F = 8000\,Hz$

Cut off frequencies $f_{c_1} = 1000\,Hz$

$f_{c_2} = 3000\,Hz$

Determine the filter coefficients for $N=7$

Soln

i) $\omega_{c_1} = 2\pi f_{c_1} T = \frac{2\pi f_{c_1}}{F} = \frac{2\pi(1000)}{8000} = \frac{\pi}{4}$

$\omega_{c_2} = 2\pi f_{c_2} T = \frac{2\pi f_{c_2}}{F} = \frac{2\pi(3000)}{8000} = \frac{3\pi}{4}$

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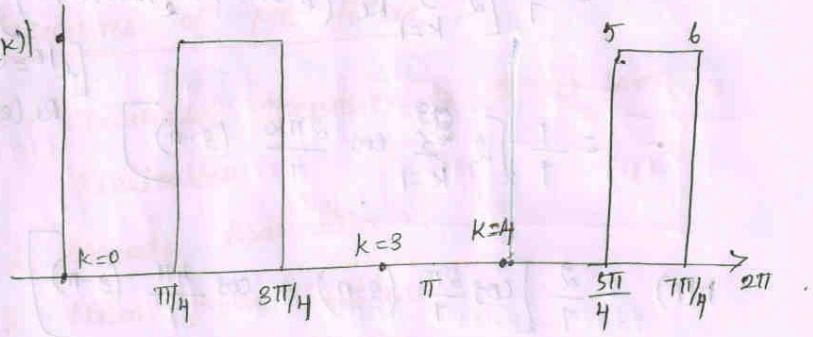

2) $H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi}{7}k}$, $k = 0, 1 \ldots 6$

$$|H(k)| = 0 \quad , k = 0, 3, 4$$
$$= 1 \quad , k = 1, 2, 5, 6 \quad \rightarrow ①$$

3) $\theta(k) = -\left(\frac{N-1}{N}\right)\pi$, $0 \le k \le \frac{N-1}{2}$ $\longrightarrow ②$

$$= -6/7\,\pi k \quad , \quad 0 \le k \le 3$$

By substituting ① & ② in 8 (or) 13

=)

$$H(k) = 0 \quad , k = 0, 3, 4$$
$$= e^{-j6\pi k/7} \quad , k = 1, 2, 5, 6$$

4) The filter coefficients are given by

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{N-1/2} Re\left(H(k)\, e^{j2\pi kn/N}\right)\right],$$

$H[0] = 0$

$$= \frac{1}{7}\left[2\sum_{k=1}^{3} Re\left(e^{-j6\pi k/7}\, e^{j2\pi kn/7}\right)\right]$$

$$\left[e^{j\theta} = \cos\theta + \ldots \quad Re(e^{j\theta}) = \ldots\right]$$

$$= \frac{1}{7}\left[2\sum_{k=1}^{3} \cos\frac{2\pi k}{7}(3-n)\right]$$

$$h(n) = \frac{2}{7}\left[\cos\frac{2\pi}{7}(3-n) + \cos\frac{4\pi}{7}(3-n)\right]$$

$\therefore h(0) = h(6) = -0.07928$

$h(1) = h(5) = -0.321$

$h(2) = h(4) = 0.11456$

$h(3) \simeq 0.57.$

HW

3) Determine the coefficients of a linear phase FIR filter of length $M = 15$ has a symmetric unit sample response and a frequency response that satisfies the conditions

$$H\left[2\pi k/15\right] = 1, \quad k = 0,1,2,3$$
$$= 0 \quad k = 4,5,6,7$$

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Topic V

Structures of FIR filters

1. Traversal Structure or Direct form Realization
2. Cascade Realization
3. Linear phase Realization
4. Poly Phase Realization

V.1 Traversal Structure or Direct form Realization :-

→ The traversal structure requires N multipliers, N-1 adders, N-1 delay elements

~~→ The Example :~~

→ The system function of a FIR filter can be written as

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= h(0) + h(1) z^{-1} + h(2) z^{-2} + \cdots h(N-1) z^{-(N-1)} \longrightarrow (1)$$

$$H(z) = \frac{Y(z)}{X(z)} = h(0) \cancel{*} h(1) z^{-1} + h(2) z^{-2} + \cdots h(N-1) z^{-(N-1)}$$

$$Y(z) = h(0) X(z) + h(1) z^{-1} X(z) + h(2) z^{-2} X(z) + \cdots h(N-1) z^{-(N-1)} X(z) \longrightarrow (2)$$

Eqn ② can be realized as in Fig 1

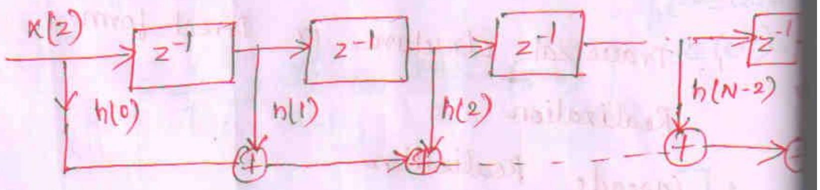

Fig 1: Direct form Realization

∴ The structure is known as Direct form Realization

V.2) Cascade Realization

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$H(z) = \frac{Y(z)}{X(z)} = h(0) + h(1) z^{-1} + h(2) z^{-2} + \dots h(N-1) z$$

$$Y(z) = h(0) X(z) + h(1) z^{-1} X(z) + h(2) z^{-2} X(z) + \dots$$
$$h(N-1) z^{-(N-1)} X(z) \longrightarrow ①$$

Eqn ① can be realized as

N odd

$$H(z) = \prod_{k=1}^{N-1/2} (b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2})$$

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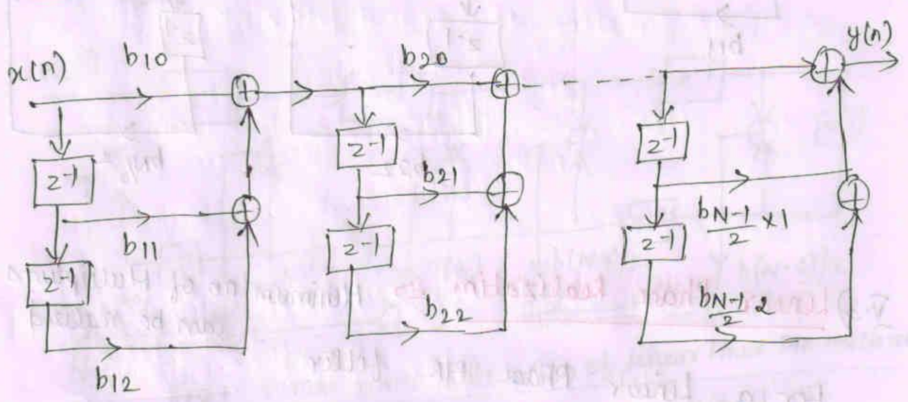
For N odd, N-1 will be even and H(z) will have (N-1)/2 second order factors,

$$H(z) = (b_{10} + b_{11} z^{-1} + b_{12} z^{-2})(b_{20} + b_{21} z^{-1} + b_{22} z^{-2}) \cdots \times (b_{\frac{N-1}{2}0} + b_{\frac{N-1}{2}1} z^{-1} \cdots)$$

Cascade Realization for N is odd



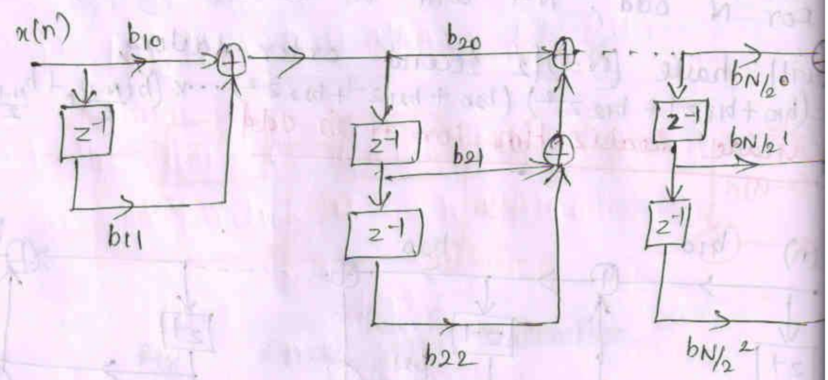
For N even :

$$H(z) = (b_{10} + b_{11} z^{-1}) \prod_{k=2}^{N/2} (b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}) \quad \rightarrow (1)$$

when N is even, N-1 is odd and H(z) will have one first order factor and $\left(\frac{N-2}{2}\right)$ second order factors.

$$\boxed{H(z) = (b_{10} + b_{11} z^{-1})(b_{20} + b_{21} z^{-1} + b_{22} z^{-2})(b_{30} + b_{31} z^{-1} + b_{32} z^{-2}) \times (b_{N/2\,0} + b_{N/2\,1} + b_{N/2\,2} z^{-2})}$$

Cascade Realization for N is even



V.3) Linear Phase Realization ⇒ Minimum no of Multipliers can be realized

For a Linear Phase FIR filter

$$h(n) = h(N-1-n) \longrightarrow (1)$$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

For N even

$$H(z) = \sum_{n=0}^{\frac{N-2}{2}} h(n) z^{-n} + \sum_{n=N/2}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) z^{-n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n) z^{-(N-1-n)} \rightarrow (2)$$

Sub Eqn (1) in (2)

$$H(z) = \sum_{n=0}^{N-2/2} h(n) z^{-n} + \sum_{n=0}^{\frac{N-2}{2}} h(n) z^{N-1-n} \rightarrow (3)$$

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$$= \sum_{n=0}^{\frac{N-2}{2}} h(n)\left[z^{-n} + z^{-(N-1-n)}\right] \rightarrow (4)$$

Eqn (4) can be realized as in Fig 1

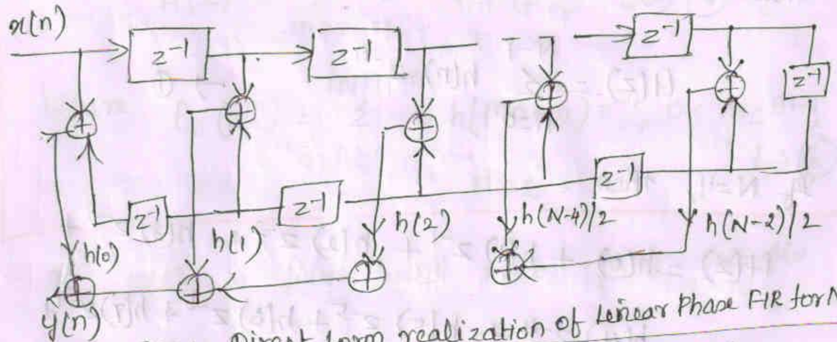

Fig 1: Direct form realization of Linear Phase FIR for N even

For N even, the no of multipliers required as $N/2$

For N odd

$$H(z) = \sum_{n=0}^{N-3/2} h(n) z^{-n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) z^{-n} + h\left(\frac{N-1}{2}\right) z^{-(N-1)/2}$$

$$= h(N-1/2)\, z^{-(N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[z^{-n} + z^{-(N-1-n)}\right] \quad \rightarrow (5)$$

Eqn (5) can be realized as in Fig 2

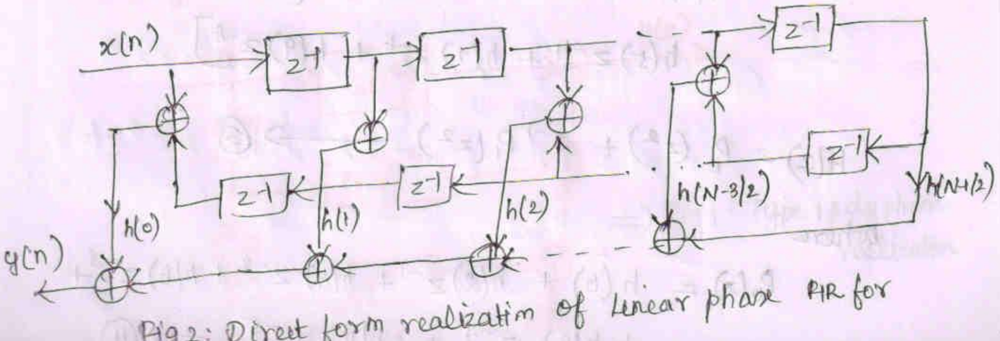

Fig 2: Direct form realization of Linear phase FIR for N odd

V.4) Poly Phase Realization of FIR filter

Let the impulse response having N coefficients, The s/m function of such a filter is gn. by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \rightarrow ①$$

If N=11, then

$$H(z) = h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5} + h(6) z^{-6} + h(7) z^{-7} + h(8) z^{-8} + h(9) z^{-9} + h(10) z^{-10} \rightarrow ②$$

② can be divided into even indexed coefficients and odd indexed coefficients.

$$H(z) = h(0) + h(2) z^{-2} + h(4) z^{-4} + h(6) z^{-6} + h(8) z^{-8} + h(10) z^{-10} + h(1) z^{-1} + h(3) z^{-3} + h(5) z^{-5} + h(7) z^{-7} + h(9) z^{-9}$$

$$= h(0) + h(2) z^{-2} + h(4) z^{-4} + h(6) z^{-6} + h(8) z^{-8} + h(10) z^{-10} + z^{-1} [h(1) + h(3) z^{-2} + h(5) z^{-4} + h(7) z^{-6} + h(9) z^{-8}]$$

$$H(z) = P_0(z^2) + z^{-1} P_1(z^2) \longrightarrow ③$$

where

$$P_0(z) = h(0) + h(2) z^{-1} + h(4) z^{-2} + h(6) z^{-3} + h(8) z^{-4} + h(10) z^{-5} \rightarrow ④$$

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$$P_1(z) = h(1) + h(3)z^{-1} + h(5)z^{-2} + h(7)z^{-3} + h(9)z^{-4} \longrightarrow (5)$$

$\therefore$ $H(z)$ can be written as

$$H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^M) \longrightarrow (5)$$

where $P_m(z^M) = \sum_{n=0}^{(N+1/M)} h(Mn+m)z^{-n}, \; 0 \le m \le M-1 \longrightarrow (6)$

$M = 2$

If $H(z)$ is partitioned into three sub sles

$H(z)$ can be written as

$$H(z) = P_0(z^3) + P_1(z^3) + P_2(z^3), \longrightarrow (7)$$

$M = 3$

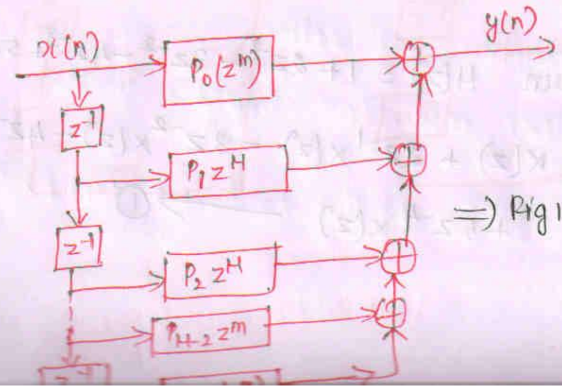
where

$$P_0(z) = h(0) + h(3)z^{-1} + h(6)z^{-2} + h(9)z^{-3}$$

$$P_1(z) = h(1) + h(4)z^{-1} + h(7)z^{-2} + h(10)z^{-3} \longrightarrow (8)$$

$$P_2(z) = h(2) + h(5)z^{-1} + h(8)z^{-2},$$

(5) can be realized as in Fig 1 and known as Type I polyphase realization



$\Rightarrow$ Fig 1: Type I polyphase realization

⇒ If we replace m by M-1-m in (5), type 2 polyphase realization is obtained

$$H(z) = \sum_{m=0}^{M-1} P_{M-1-m}(z^M)$$

$$= \sum_{m=0}^{M-1} Q_m(z^M) \longrightarrow (9)$$

where

$$Q_m(z^M) = P_{M-1-m}(z^M)$$

⇒ If we replace m by -m in (6), type 3 polyphase realization is obtained

$$H(z) = \sum_{m=0}^{M-1} z^m R_m(z^M) \longrightarrow (10)$$

where

$$R_0(z^M) = P_0(z^M) \text{ \&}$$

$$R_m(z^M) = z^{-1} P_{M-m}(z^M)$$

Problems on Structure of FIR Filters

1) Determine the direct form realization of system function $H(z) = 1+2z^{-1}-3z^{-2}-4z^{-3}+$

Soln

Given $H(z) = 1+2z^{-1}-3z^{-2}-4z^{-3}+5z^{-4}$

$$Y(z) = X(z) + 2z^{-1}X(z) - 3z^{-2}X(z) - 4z^{-3}X(z)$$
$$+ 5z^{-4}X(z) \longrightarrow (1)$$

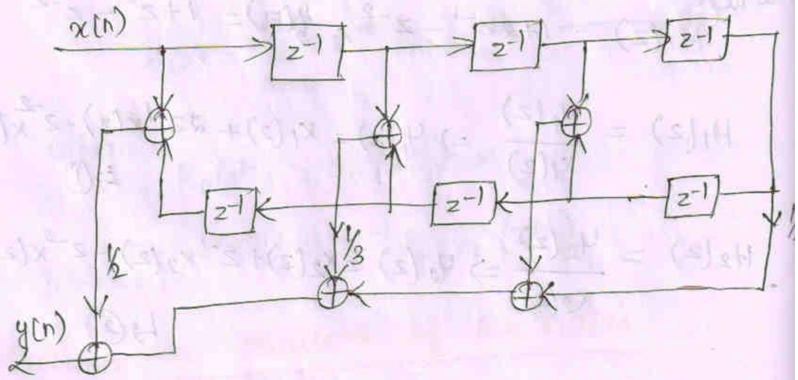
3) Realize the system function

$$H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} +$$

Soln

The given system function $H(z)$ is that of a linear phase FIR filter

$h(n) = h(N-1-n)$ and realization is given in Fig1



4) Realize the s/m function

$$H(z) = 1 + 4z^{-1} - 3z^{-2} + 6z^{-3} - 9z^{-4} + 5z^{-5} +$$

using polyphase realization

Divide $H(z)$ into two branches yields

$$H(z) = 1 - 3z^{-2} - 9z^{-4} + 7z^{-6} + z^{-1}(4 + 6z^{-2} + 5$$

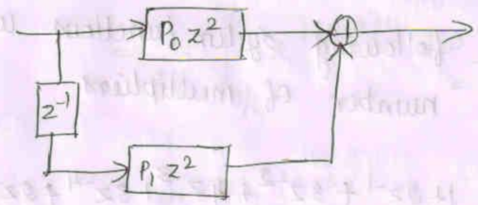
(71)

$$H(z) = P_0(z^2) + z^{-1} P_1(z^2)$$

where

$$\left.\begin{aligned} P_0(z^2) &= 1 - 3z^{-2} - 9z^{-4} + 7z^{-6} \\ P_1(z^2) &= 4 + 6z^{-2} + 5z^{-4} \end{aligned}\right\} \rightarrow (1)$$

The polyphase realization of eqn (1) can be given as



$P_0(z^2)$ and $P_1(z^2)$ can be realized in direct form and corresponding realization of $H(z)$ as in eqn (1) can be gn as

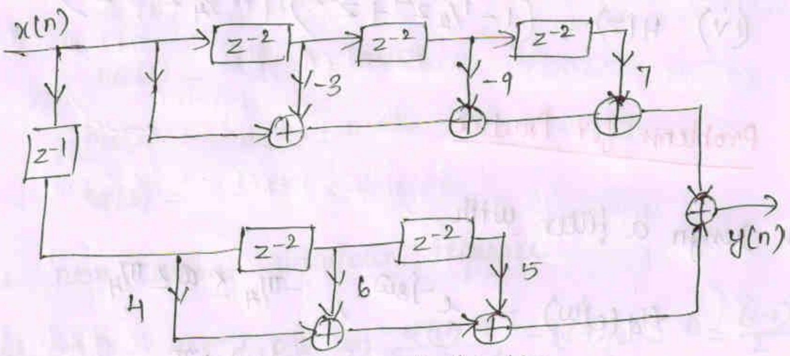


Fig1: Polyphase Realization

This can be realized by reducing the delay elements and is known as Canononical polyphase realization

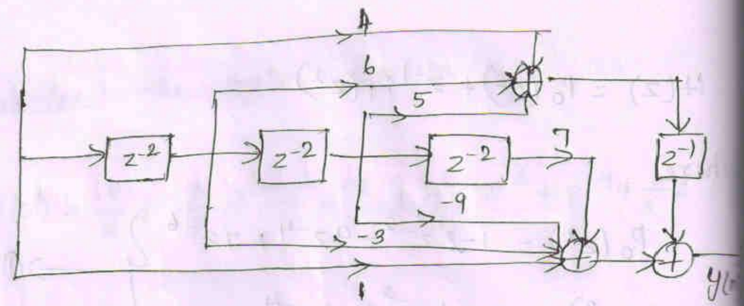


Fig 2: Canonical polyphase realization

Problems for HW

1) Realize the following system functions using a minimum number of multipliers

(i) $H(z) = 1+2z^{-1}+3z^{-2}+4z^{-3}+3z^{-4}+2z^{-5}+z^{-6}$

(ii) $H(z) = 1+\frac{1}{3}z^{-1}+\frac{1}{3}z^{-2}+z^{-3}$

(iii) $H(z) = 1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}+z^{3}$

(iv) $H(z) = \left(1-\frac{1}{2}z^{-1}+z^{-2}\right)\left(1-\frac{1}{4}z^{-1}+z^{2}\right)$

Problems for Practice

1) Design a filter with

$$H_d(e^{j\omega}) = e^{-j3\omega}, \quad -\pi/4 \le \omega \le \pi/4$$
$$= 0, \quad \pi/4 < |\omega| \le \pi$$

Using a Hamming window with $N=7$

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Given

$$H_d(e^{j\omega}) = e^{-j3\omega}$$

freq response $e^{-j\omega(N-1)/2} \Rightarrow h(n)$ symmetrical about $n = \frac{N-1}{2} = 3$ $\therefore$ causal sequence is obtained

$$\therefore h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j3\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j(n-3)\omega}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j(n-3)\pi/4} - e^{-j(n-3)\pi/4}}{j(n-3)}\right]$$

$$= \frac{\sin \pi/4\,(n-3)}{\pi(n-3)}$$

For $N = 7$,

$$h_d(0) = h_d(6) = 0.075$$

$$h_d(1) = h_d(5) = 0.159$$

$$h_d(2) = h_d(4) = 0.22$$

$$h_d(3) = 0.25$$

The non causal window sequence

$$w_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1}, \quad -(N-1)/2 \le n \le \frac{(N-1)}{2}$$

$$= 0, \quad \text{otherwise}$$

For $N=7$

$$w_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1}, \quad -3 \le n \le 3$$
$$= 0, \text{ otherwise}$$

$$w_{Hn}(0) = 0.5 + 0.5 = 1$$

$$w_{Hn}(-1) = w_{Hn}(1) = 0.5 + 0.5\cos\pi/3 = 0.75$$

$$w_{Hn}(-2) = w_{Hn}(2) = 0.5 + 0.5\cos 2\pi/3 = 0.25$$

$$w_{Hn}(-3) = 0.5 + 0.5\cos\pi = 0$$

$\therefore$ The causal window seq can be obtained by shifting $w_{Hn}(n)$ to right by 3 samples

$$w_{Hn}(0) = w_{Hn}(6) = 0; \quad w_{Hn}(1) = w_{Hn}(5) = 0.2$$

$$w_{Hn}(2) = w_{Hn}(4) = 0.75 \text{ \& } w_{Hn}(3) = 1$$

2) Obtain cascade realization with minimum no. of multipliers for the s/m function

$$H(z) = \left(\frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2}\right)\left(1 + \frac{1}{3}z^{-1} + z^{-2}\right)$$

Soln

It is given that $H(z)$ is a product of factors that have the linear phase symmetry property. The corresponding cascade realization is given as in Fig shown below

* IMPORTANT 2 MARKS

1) Specifications and Desired Impulse Response (FIR FILTER BY FS)

Type Filter	Specifications	Impulse Response
Low-Pass	$H_d(e^{j\omega}) = \begin{cases} 1; & -\omega_c \le \omega \le +\omega_c \\ 0; & -\pi \le \omega < -\omega_c \\ 0, & \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$ $= \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{j\omega n}\, d\omega$
High Pass	$H_d(e^{j\omega}) = \begin{matrix} 1, & -\pi \le \omega \le -\omega_c \\ 1, & \omega_c \le \omega \le \pi \\ 0; & -\omega_c < \omega < +\omega_c \end{matrix}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n} d\omega$ $= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega n} d\omega$
Band pass	$H_d(e^{j\omega})$ $\begin{matrix} 1 & ; -\omega_{c_2} \le \omega \le -\omega_{c_1} \\ 1 & ; +\omega_{c_1} \le \omega < \omega_{c_2} \\ 0; & -\pi \le \omega < -\omega_{c_2} \\ 0; & -\omega_{c_1} < \omega < +\omega_{c_1} \\ 0; & \omega_{c_2} < \omega \le \pi \end{matrix}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n} d\omega$ $= \frac{1}{2\pi}\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c_1}}^{\omega_{c_2}} e^{j\omega n} d\omega$
	$1; \; -\pi \le \omega \le -\omega_{c_2}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n} d\omega$

2) Give the equation of the frequency sampling realization of FIR filter

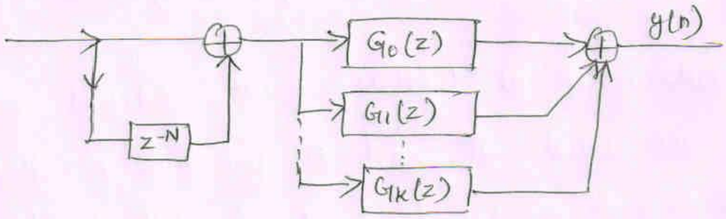
The z transform of FIR filter designed using frequency sampling method is gn by

$$H(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} G_k(z)$$

where $G_k(z) = \dfrac{H(k)}{1 - z^{-1} W^k}$

where $W = e^{j2\pi/N}$

Realizatim can be given by



3) Compare FIR and IIR Filter

FIR Filter	IIR Filter
1. Impulse response is restricted to finite no of samples	Impulse response extends over an infinite duratim
2. This have precisely Linear phase	These filters do not have linear phase
3. Closed form design equations do not exist	A variety of frequency selective filters can be designed using closed form design formulas

UNIT I – INTRODUCTION

Syllabus

Introduction to Digital Signal Processing: Discrete time signals & sequences, linear shift invariant systems, stability, and causality. Linear constant coefficient difference equations. Frequency domain representation of discrete time signals and systems.

Introduction

Signal

A signal is any physical quantity that carries information, and that varies with time, space, or any other independent variable or variables. Mathematically, a signal is defined as a function of one or more independent variables.

1 – Dimensional signals mostly have time as the independent variable. For example,

Eg., $S_1(t) = 20\,t^2$

2 – Dimensional signals have two independent variables. For example, image is a 2 – D signal whose independent variables are the two spatial coordinates (x,y)

Eg., $S_2(t) = 3x + 2xy + 10y^2$

Video is a 3 – dimensional signal whose independent variables are the two spatial coordinates, (x,y) and time (t).
Similarly, a 3 – D picture is also a 3 – D signal whose independent variables are the three spatial coordinates (x,y,z).

Signals $S_1(t)$ and $S_2(t)$ belong to a class that are precisely defined by specifying the functional dependence on the independent variables.
Natural signals like speech signal, ECG, EEG, images, videos, etc. belong to the class which cannot be described functionally by mathematical expressions.

System

A system is a physical device that performs an operation on a signal. For example, natural signals are generated by a system that responds to a stimulus or force.
For eg., speech signals are generated by forcing air through the vocal cords. Here, the vocal cord and the vocal tract constitute the system (also called the vocal cavity). The air is the stimulus.
The stimulus along with the system is called a signal source.

An electronic filter is also a system. Here, the system performs an operation on the signal, which has the effect of reducing the noise and interference from the desired information – bearing signal.

When the signal is passed through a system, the signal is said to have been processed.

Processing

The operation performed on the signal by the system is called **Signal Processing.** The system is characterized by the type of operation that it performs on the signal. For example, if the operation is linear, the system is called linear system, and so on.

Digital Signal Processing

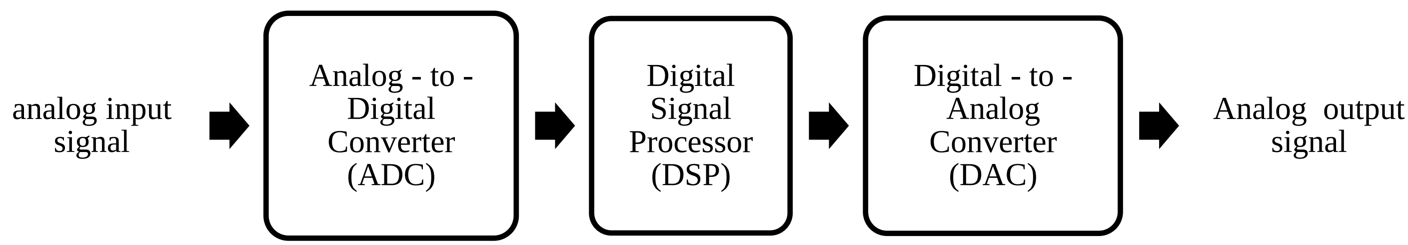
Digital Signal Processing of signals may consist of a number of mathematical operations as specified by a software program, in which case, the program represents an implementation of the system in software. Alternatively, digital processing of signals may also be performed by digital hardware (logic circuits). So, a digital system can be implemented as a combination of digital hardware and software, each of which performs its own set of specified operations.

Basic elements of a Digital Signal Processing System

Most of the signals encountered in real world are analog in nature .i.e., the signal value and the independent variable take on values in a continuous range. Such signals may be processed directly by appropriate analog systems, in which case, the processing is called **analog signal processing**. Here, both the input and output signals are in analog form.

These analog signals can also be processed digitally, in which case, there is a need for an interface between the analog signal and the **Digital Signal Processor**. This interface is called the **Analog – to – Digital Converter (ADC)**, whose output is a digital signal that is appropriate as an input to the digital processor.
In applications such as speech communications, that require the digital output of the digital signal processor to be given to the user in analog form, another interface from digital domain to analog domain is required. This interface is called the **Digital – to – Analog Converter (DAC)**.
In applications like radar signal processing, the information extracted from the radar signal, such as the position of the aircraft and its speed are required in digital format. So, there is no need for a DAC in this case.

Block Diagram Representation of Digital Signal Processing



Advantages of Digital Signal Processing over Analog Signal Processing

1. A digital programmable system allows flexibility in reconfiguring the digital signal processing operations simply by changing the program.
   Reconfiguration of an analog system usually implies a redesign of the hardware followed by testing and verification.
2. Tolerances in analog circuit components and power supply make it extremely difficult to control the accuracy of analog signal processor.
   A digital signal processor provides better control of accuracy requirements in terms of word length, floating – point versus fixed – point arithmetic, and similar factors.
3. Digital signals are easily stored on magnetic tapes and disks without deterioration or loss of signal fidelity beyond that introduced in A/D conversion. So the signals become transportable and can be processed offline.
4. Digital signal processing is cheaper than its analog counterpart.
5. Digital circuits are amenable for full integration. This is not possible for analog circuits because inductances of respectable value (μH or mH) require large space to generate flux.
6. The same digital signal processor can be used to perform two operations by time multiplexing, since digital signals are defined only at finite number of time instants.

7. Different parts of digital signal processor can work at different sampling rates.
8. It is very difficult to perform precise mathematical operations on signals in analog form but these operations can be routinely implemented on a digital computer using software.
9. Several filters need several boards in analog signal processing, whereas in digital signal processing, same DSP processor is used for many filters.

<u>Disadvantages of Digital Signal Processing over Analog Signal Processing</u>

1. Digital signal processors have increased complexity.
2. Signals having extremely wide bandwidths require fast – sampling – rate ADCs. Hence the frequency range of operation of DSPs is limited by the speed of ADC.
3. In analog signal processor, passive elements are used, which dissipate very less power.
   In digital signal processor, active elements like transistors are used, which dissipate more power.

The above are some of the advantages and disadvantages of digital signal processing over analog signal processing.

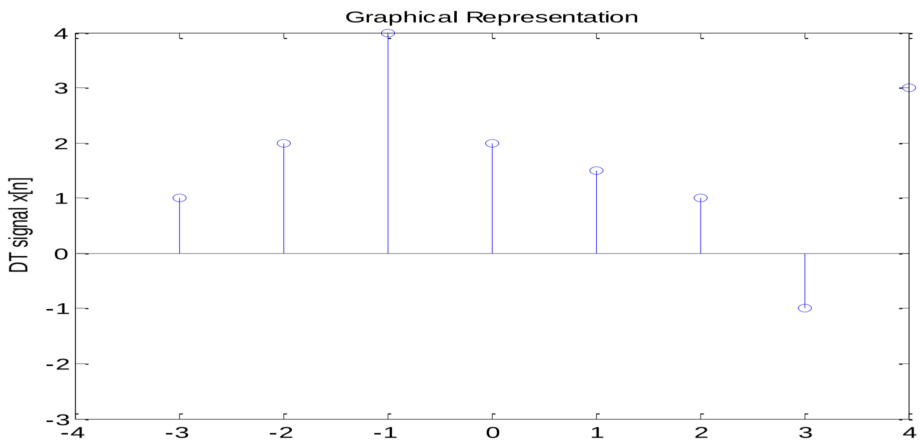
<u>Discrete – time signals</u>

A discrete time signal is a function of an independent variable that is an integer, and is represented by x [n] , where n represents the sample number (**and not the time at which the sample occurs).**
A discrete time signal is not defined at instants between two successive samples, or in other words, for non – integer values of n. (**But, it is not zero, if n is not an integer).**

<u>Discrete time signal representation</u>

The different representations of a discrete time signal are

1. **<u>Graphical Representation</u>**



2. **<u>Functional representation</u>**

$$x[n] = \begin{cases} 1, for\ n = 1, 2\ 3 \\ 4, for\ n = 2 \\ 0, elsewhere \end{cases}$$

3. **<u>Tabular representation</u>**

N	-	- - - - -	-2	-1	0	1	2	3	4	5	- - - - - -
x [n]	-	- - - - -	0	0	1	1	4	1	0	0	- - - - - -

4. **Sequence representation**

x [n] = { - , -, -. -, - , 0, 0, 1, 4, 1, 0, 0, - , - , - , - }

the above is a representation of a two – sided infinite duration sequence, and the symbol $\uparrow$ indicates the time origin (n = 0).

If the sequence is zero for n < 0, it can be represented as

x [n] = { 1, 4, 1, 2, - , - , - , - }

Here the leftmost point in the sequence is assumed to be the time origin, and so the symbol $\uparrow$ is optional in this case.

A finite duration sequence can be represented as
x[n] = { 3, -1, -2, 5, 0, 4, -1}

This is referred to as a 7 – point sequence.

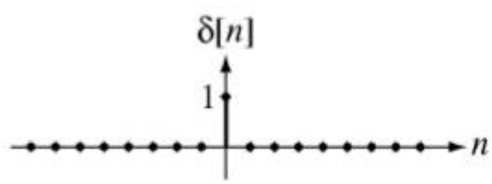
Elementary discrete time sequences

These are the basic sequences that appear often, and play an important role. Any arbitrary sequence can be represented in terms of these elementary sequences.

1. **Unit – Sample sequence** It is denoted by δ [n]. It is defined as

$$\delta[n] = \begin{cases} 1, for\ n = 0 \\ 0, for\ n \neq 0 \end{cases}$$

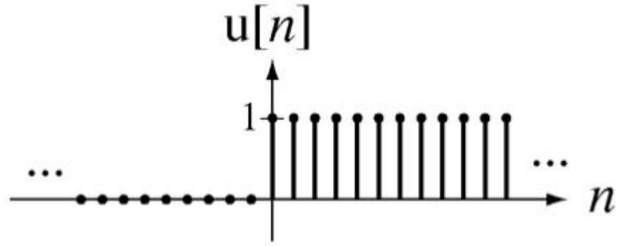
It is also referred as discrete time impulse.
It is mathematically much less complicated than the continuous impulse δ (t), which is zero everywhere except at t = 0. At t = 0, it is defined in terms of its area (unit area), but not by its absolute value.
It is graphically represented as



2. **Unit step sequence**
It is denoted by u [n] and defined as

$$u[n] = \begin{cases} 1, for\ n \geq 0 \\ 0, for\ n < 0 \end{cases}$$

It is graphically represented as

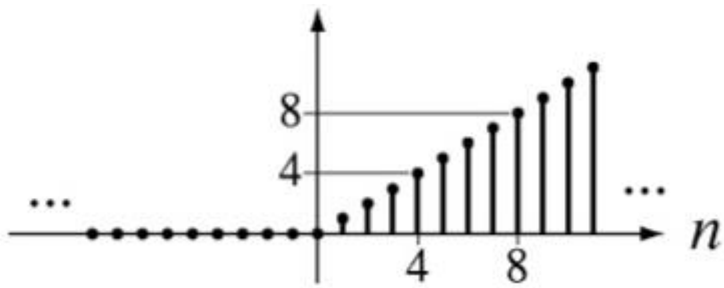


3. Unit ramp sequence

It is denoted by U_r [n], and is defined as

$$u_r[n] = \begin{cases} n, for\ n \geq\ 0 \\ 0, for\ n < 0 \end{cases}$$

It is graphically represented as



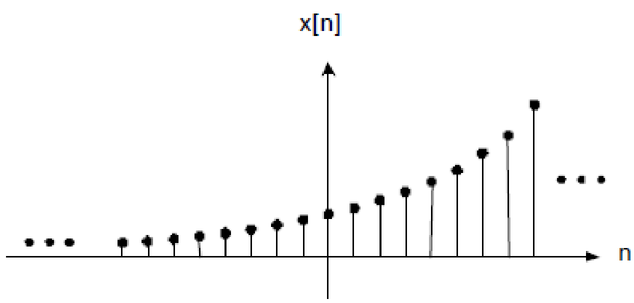
4. Exponential sequence

It is defined as

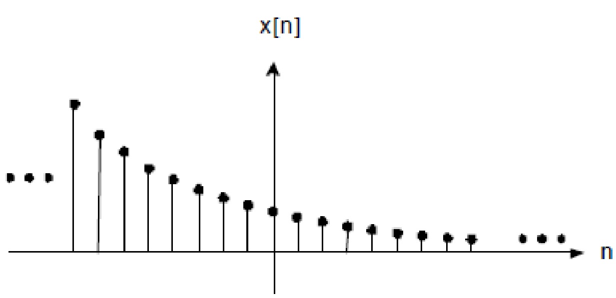
$$x[n] = a^n\ for\ all\ n$$

a. If a is real, x[n] is a real exponential.

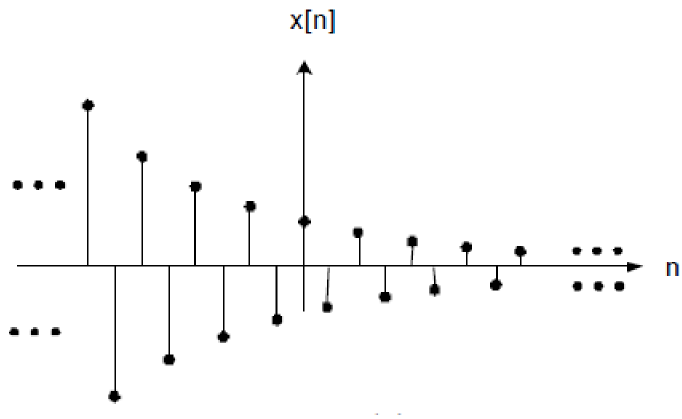
a > **1**



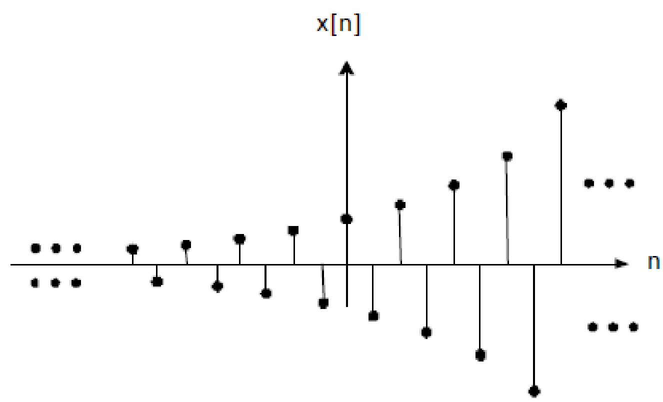
a < **1**



-1 < a < 0



a < -1



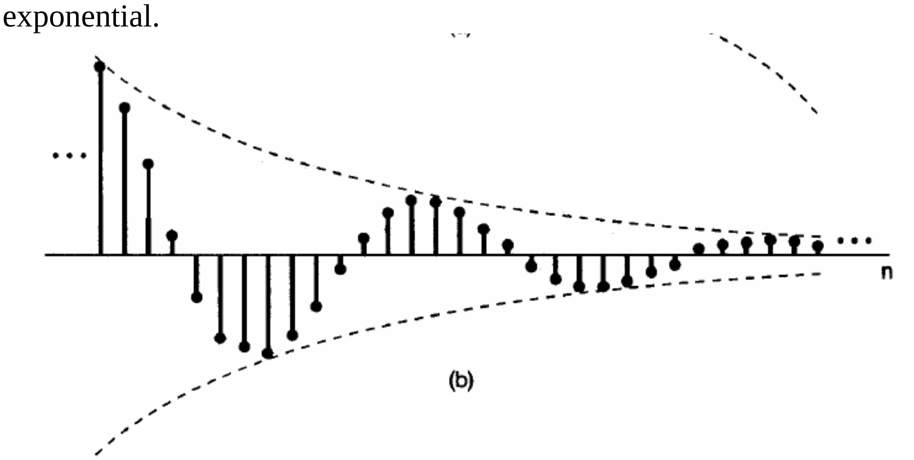
b. If a is complex valued, then a can be expressed as a = $re^{j\theta}$, so that x[n] can be represented as

$$\begin{aligned} x[n] &= r^n e^{jn\theta} \\ &= r^n[\cos n\theta + j\sin n\theta] \end{aligned}$$

So, x [n] is represented graphically by plotting the real part and imaginary parts separately as functions of n, which are

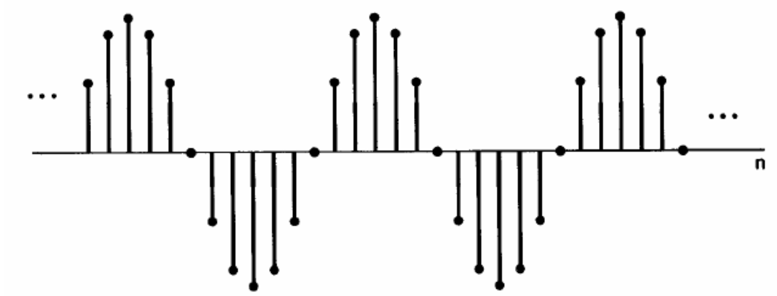
$$x_R[n] = r^n \cos n\theta$$
$$x_I[n] = r^n \sin n\theta$$

If r < 1, the above two functions are damped cosine and sine functions, whose amplitude is a decaying exponential.

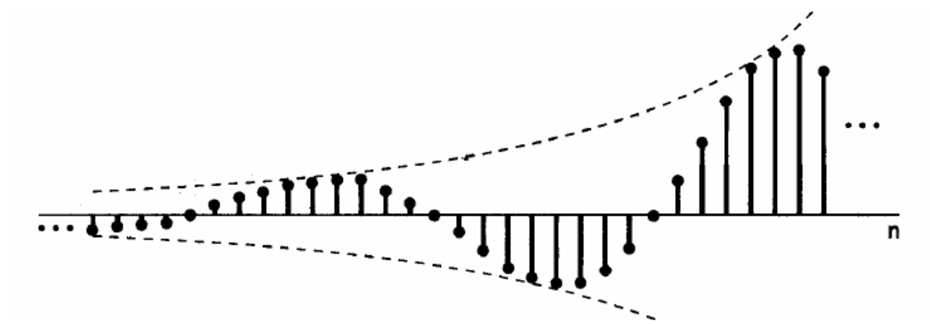


(b)

If r = 1, then both the functions have fixed amplitude of unity.



If r > 1, then they are cosine and sine functions respectively, with exponentially growing amplitudes.

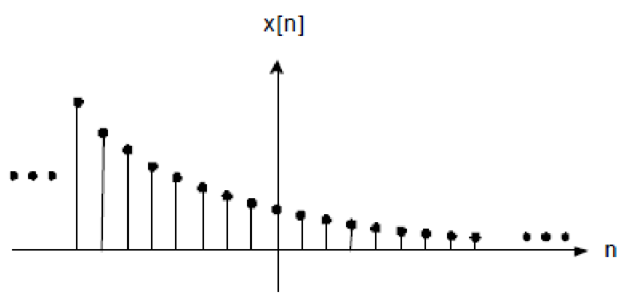


Alternatively, x [n] can be represented by the amplitude and phase functions:

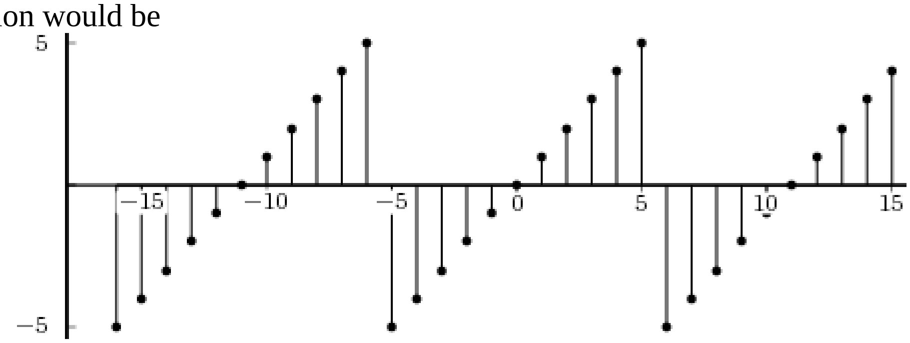
Amplitude function, $A[n] = |x[n]| = r^n$

Phase function, $\emptyset[n] = \angle x[n] = n\theta$

For example, for r < 1, the amplitude function would be



And the phase function would be



Although the phase function $\emptyset[n] = n\theta$ is a linear function of n, it is defined only over an interval of 2π (since it is an angle).i.e., over an interval –π<θ<π or 0<θ<2π.

So we subtract multiples of 2π from $\emptyset[n]$ before plotting .i.e., we plot $\emptyset[n]$ modulo 2π instead of $\emptyset[n]$. This results in a piecewise linear graph for the phase function, instead of a linear graph.

Classification of Discrete – Time Sequences:

1. **Energy Signals and Power Signals**

   The energy of a signal x[n] is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

   If this energy is finite, i.e., $0 < E < \infty$, then x[n] is called an **Energy Signal**.

   For signals having infinite energy, the average power can be calculated, which is defined as

$$P_{av} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$or, P_{av} = \lim_{N\to\infty} \frac{1}{2N+1} E_N , \qquad where$$

   E_N = signal energy of x[n] over the finite interval $-N \le n \le N$, .i.e.,

$$E = \lim_{N\to\infty} E_N$$

- For signals with finite energy .i.e., for Energy Signals, E is finite, thus resulting in zero average power. So, for energy signals, $P_{av} = 0$.
- Signals with infinite energy may have finite or infinite average power. If the average power is finite and nonzero, such signals are called **Power Signals**.
- Signals with finite power have infinite energy.
- If both energy, E as well as average power, P_{av} of a signal are infinite, then the signal is neither an energy signal nor a power signal.
- Periodic signals have infinite energy. Their average power is equal to its average power over one period.
- A signal cannot both be an energy signal and a power signal.
- All practical signals are energy signals.

2. **Periodic and aperiodic signals**

   A signal x[n] is periodic with period N if and only if

$$x[n+N] = x[n] \; \forall \; n$$

   The smallest N for which the above relation holds is called the **fundamental period.**
   If no finite value of N satisfies the above relation, the signal is said to be **aperiodic** or **non – periodic.**
   The sum of M periodic Discrete – time sequences with periods N_1, N_2, …, N_M, is always periodic with period N where

$$N = LCM(N_1, N_2, \dots, N_M)$$

3. **Even and Odd Signals**

   A real – valued discrete – time signal is called an **Even Signal** if it is identical with its reflection about the origin .i.e., it must be symmetrical about the vertical axis.

$$x[n] = x[-n] \; \forall n$$

A real – valued discrete – time signal is called an **Odd Signal** if it is antisymmetrical about the vertical axis.

$$x[n] = -x[-n] \;\; \forall n$$

From the above relation, it can be inferred that an odd signal must be zero at time origin, n = 0.

Every signal x[n] can be expressed as the sum of its even and odd components.

$$x[n] = x_e[n] + x_o[n]$$

Where

$$x_e[n] = \frac{x[n] + x[-n]}{2}$$

$$x_o[n] = \frac{x[n] - x[-n]}{2}$$

- Product of even and odd sequences results in an odd sequence.
- Product of two odd sequences results in an even sequence.
- Product of two even sequences results in an even sequence.

4. <u>Conjugate Symmetric and Conjugate Antisymmetric sequences</u>

A complex discrete – time signal is **conjugate – symmetric** if

$$x[n] = x^*[-n] \;\; \forall n$$

And **conjugate – antisymmetric** if

$$x[n] = -x^*[-n] \;\; \forall n$$

Any complex signal can be expressed as the sum of conjugate – symmetric and conjugate – antisymmetric parts

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

Where

$$x_{cs}[n] = \frac{x[n] + x^*[-n]}{2}$$

And

$$x_{ca}[n] = \frac{x[n] - x^*[-n]}{2}$$

5. <u>Bounded and Unbounded sequences</u>

A discrete – time sequence x[n] is said to be **bounded** if each of its samples is of finite magnitude .i.e.,

$$|x[n]| \leq M_x < \infty \;\; \forall n$$

For example,

The unit step sequence u[n] is a bounded sequence,
but the sequence nu[n] is an unbounded sequence.

6. <u>Absolutely summable and square summable sequences</u>

A discrete – time sequence x[n] is said to be **absolutely summable** if,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

And it is said to be **square summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \quad (\boldsymbol{Energy\ Signal})$$

<u>Discrete – Time Systems</u>

A system accepts an input such as voltage, displacement, etc. and produces an output in response to this input. A system can be viewed as a process that results in transforming input signals into output signals.

A discrete – time system can be represented as

$$x[n] \;\rightarrow\; y[n]$$
$$or, \qquad y[n] = T\,\{x[n]\}$$

<u>Discrete – Time System Properties</u>

1. **<u>Linearity</u>**

A system is said to be **linear** if it satisfies superposition principle, which in turn is a combination of **additivity** and **homogeneity**.

Additivity implies that

If the response of the DT system to $x_1[n]$ is $y_1[n]$, and the response to $x_2[n]$ is $y_2[n]$, then the response of the system to $\{x_1[n]+x_2[n]\}$ must be $\{y_1[n]+y_2[n]\}$.

Homogeneity implies that

if the response of a DT system to x[n] is y[n], then the response of the system to ax[n] must be ay[n], where a is a constant.

Thus, for a DT system,

If

$$x[n] \;\rightarrow\; y[n]$$
$$x_1[n] \;\rightarrow\; y_1[n]$$
$$and, \qquad x_2[n] \;\rightarrow\; y_2[n]$$

Then according to **additivity principle**

$$x_1[n] + x_2[n] \;\rightarrow\; y_1[n] + y_2[n]$$

And according to **homogeneity principle**

$$ax[n] \;\rightarrow\; ay[n] \;\; (a = constant)$$

- If a = 0, then the above relation implies that a zero input must result in a zero output.

Combining the above two principle to get **superposition principle**, we obtain

A system is Linear if it satisfies the following relation

$$ax_1[n] + bx_2[n] \;\rightarrow\; ay_1[n] + by_2[n] \; (a, b = constants)$$

2. **Time – Variant and Time – Invariant Systems**
A system is **time – invariant** if its characteristics and behavior are fixed over time .i.e., a time – shift in input signal causes an identical time – shift in output signal.

$$if\ x[n] \ \rightarrow \ y[n]$$
$$then, x[n-n_0] \ \rightarrow y[n-n_0] \ \forall\, n_0$$

If the above the relation is not satisfied, then the system is **time – variant**.

3. **Causal and Non – causal Systems**
A system is **causal or non – anticipatory or physically realizable**, if the output at any time n_0 depends only on present and past inputs ($n \leq n_0$), but not on future inputs.
In other words, if the inputs are equal upto some time n_0, the corresponding outputs must also be equal upto that time n_0, for a **causal system.**

4. **Stable and unstable systems**
A **stable system** is one in which, a bounded input results in a response that does not diverge. Then the system is said to be **BIBO stable**.
For a system, if the input is bounded .i.e,

$$if \ \ |x[n]| \leq M_x < \infty \ \ \forall n$$

And if the corresponding output is also bounded .i.e.,

$$|y[n]| \leq M_y < \infty \ \ \forall n$$

Then the system is said to be **BIBO stable.**

5. **Memory and memoryless systems**
A system is said to possess **memory**, or is called a **dynamic system**, if its output depends on past or future values of the input.
If the output of the system depends only on the present input, the system is said to be **memoryless.**

6. **Invertible systems**
A system is said to be **invertible** if by observing the output, we can determine its input. i.e., we can construct an inverse system that when cascaded with the given system, yields an output equal to the original input.
A system can have inverse if distinct inputs lead to distinct outputs.

7. **Passive and lossless systems**
A system is said to be **passive** if the output y[n] has at most the same energy as the input.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \sum_{n=-\infty}^{\infty} |y[n]|^2 < \infty$$

If the energy of the output is equal to the energy of the input, then the system is said to be **lossless.**

Properties of Unit Impulse Sequence

Multiplication property

When a sequence x[n] is multiplied by a unit impulse located at k i.e., δ[n-k], picks out a single value/sample of x[n] at the location of the impulse i.e., x[k].

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$
$$= impulse\ with\ strength\ x[k] located\ at\ n = k$$

Sifting property

The impulse function δ[n-k] "sifts" through the function x[n] and pulls out the value x[k]

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-k] = x[k]$$

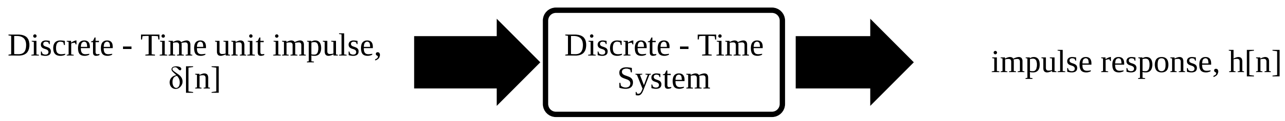
Signal decomposition

Any arbitrary sequence x[n] can be expressed as a weighted sum of shifted impulses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Impulse response

Impulse response of a discrete – time system is defined as the output/response of the system to unit impulse input and is represented by **h[n]**.



If for a system,

$$x[n] \;\rightarrow y[n]$$

Then,

$$\delta[n] \;\rightarrow\; h[n]$$

If the DT system satisfies the property of time – invariance, then,

$$\delta[n-k] \rightarrow h[n-k]$$

In addition to being time – invariant, if the system also satisfies linearity (homogeneity and additivity), then,

Homogeneity:

$$x[k]\delta[n-k] \rightarrow x[k]h[n-k]$$

Additivity:

$$\sum_{k=-\infty}^{\infty} \delta[n-k] \rightarrow \sum_{k=-\infty}^{\infty} h[n-k]$$

Combining the above two properties, a **Linear Time – Invariant (LTI)** System can be described by the input – output relation by

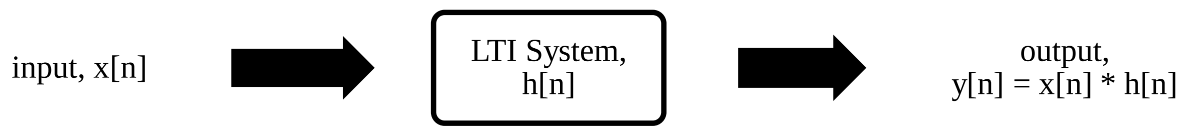
$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The Left hand side is the input x[n] expressed as a weighted sum of shifted impulses (from signal decomposition property of impulse function). So, the right hand side must be the output y[n] of the DT system in response to input x[n].

Thus the output of a **Linear Time – Invariant (LTI) system** can be expressed as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$or, y[n] = x[n] * h[n]$$

The above relation is called **Convolution Sum.**

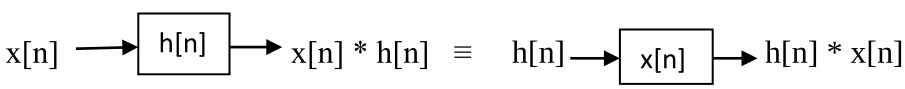


So, the impulse response h[n] of an LTI DT system completely characterizes the system .i.e., a knowledge of h[n] is sufficient to obtain the response of an LTI system to any arbitrary input x[n].

<u>Properties of Convolution Sum</u>

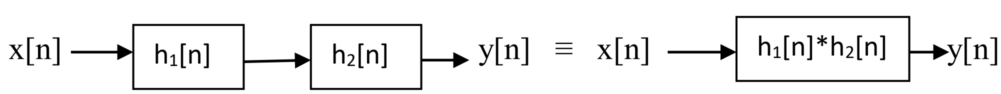
1. **<u>Commutative Property</u>**

$$x[n] \; * \; h[n] = h[n] \; * \; x[n]$$



2. **<u>Associative Property</u>**

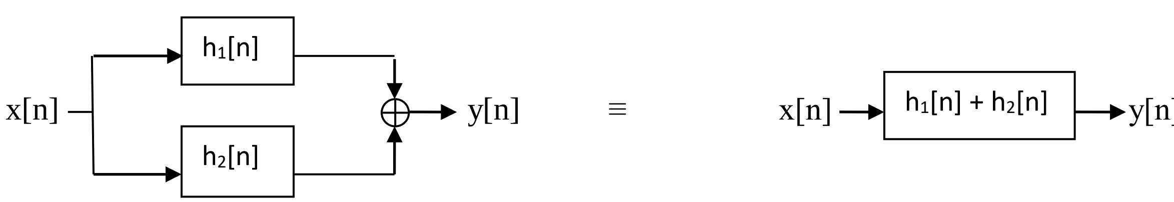
$$x[n] * \{h_1[n] * \; h_2[n]\} \; = \{x[n] * \; h_1[n]\} * h_2[n]$$



From this property it can be inferred that, a cascade combination of LTI systems can be replaced by a single system whose impulse response is the convolution of the individual impulse responses.

3. **<u>Distributive Property</u>**

$$x[n] * \; \{h_1[n] + \; h_2[n]\} = \{x[n] * \; h_1[n]\} + \; \{x[n] * \; h_2[n]\}$$



From this property, it can be inferred that, a parallel combination of LTI systems can be replaced by a single system whose impulse response is the sum of individual responses.

Relation between LTI system properties and impulse response

Memory

For an LTI system to be memoryless, the impulse response must be zero for nonzero sample positions.

$$h[n] = 0 \; for \; n \neq 0$$

$$h[n] = k\,\delta\,[n] \;\; where \; k = constant$$

Causality

For an LTI system to be causal, its impulse response must be zero for negative time instants.

$$h[n] = 0 \; for \; n < 0$$

So, for a causal LTI system the output (from the convolution sum equation) can be expressed as

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$or, y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

Stability

An LTI system is BIBO stable if its impulse response is absolutely summable.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Invertibility

An LTI system with impulse response h[n] is invertible if we can design another LTI system with impulse response $h_I[n]$ such that

$$h[n] * h_I[n] = \delta\,[n]$$

LTI systems characterized by Linear Constant – Coefficient Difference Equations (LCCDE)

In general, any LTI system with input x[n] and output y[n] can be described by an LCCDE as follows

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \;\; , \qquad a_0 \equiv 1$$

$$or, y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{l=0}^{M} b_l x[n-l]$$

Where N is called the **order** of the difference equation/ system.

This equation expresses the output of an LTI system at time n in terms of present and past inputs and past outputs.

Solution of LCCDE (Direct Solution – Solution in time domain)

Given an LCCDE, the goal is to determine the output y[n], n$\geq$ 0 given a specific input x[n], n $\geq$ 0, and a set of initial conditions.

The total solution of the LCCDE is assumed to be the sum of two parts:

Homogeneous/complementary solution, $y_H[n]$ and

Particular solution, $y_P[n]$

Homogeneous Solution

The homogeneous difference equation is obtained by substituting input x[n]=0 in the LCCDE.

$$\sum_{k=0}^{N} a_k y[n-k] = 0 ------Eq.1$$

The solution to this homogeneous equation is assumed to be in the form of an exponential .i.e.,

$$y_h[n] = \lambda^n ------Eq.2$$

Substituting Eq. 2 in Eq. 1, we obtain

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0 \ \ , a_0 = 1$$

Expanding this equation

$$\lambda^{n-N}(\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots\cdots + a_{N-1}\lambda + a_N) = 0$$

The polynomial in the parenthesis is called the characteristic polynomial of the system.

The characteristic equation is given by

$$\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots\cdots + a_{N-1}\lambda + a_N = 0$$

Its solution has N roots denoted by λ_1, λ_2, …, λ_N, which can be real or complex.

Complex valued roots occur as complex conjugate pairs.

If some roots are identical, then we have multiple order roots.

If all roots are distinct, then the general solution is given by

$$y_H[n] = C_1{\lambda_1}^n + C_2{\lambda_2}^n + \cdots\cdots + C_N{\lambda_N}^n$$

C_1, C_2, … , C_N are weighting coefficients.

For multiple order roots, if λ_1 repeats m times, then the solution is given by

$$y_H[n] = C_1{\lambda_1}^n + C_2 n{\lambda_1}^n + C_3 n^2{\lambda_1}^n \cdots\cdots + C_m n^{m-1}{\lambda_1}^n + C_{m+1}{\lambda_2}^n + C_{m+2}{\lambda_3}^n + \cdots C_N{\lambda_N}^n$$

Particular solution

The particular solution must satisfy the LCCDE for the specific input signal x[n], $n \geq 0$.

We assume a form for $y_P[n]$ that depends on the form of the input x[n] as follows

Input, x[n]	**Particular solution, $y_P[n]$**
Constant, A	Constant, K
$A M^n$	KM^n
An^M	$K_0n^M + K_1n^{M-1}+\ldots+K_M$
$A^n n^M$	$A^n(K_0n^M+K_1n^{M-1}+\ldots+K_M)$
$A \cos \omega_0 n$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$
$A \sin \omega_0 n$	

If the particular solution, y_P [n] has the same form as the homogeneous solution y_H [n], we multiply $y_P[n]$ with n or n^2 or n^3 so that it is different from $y_H[n]$.

Total solution $y[n] = y_H[n]+y_P[n]$

The total solution will contain $\{C_i\}$s from the homogeneous solution. They are determined by substituting the given initial conditions in the total solution.

Frequency domain representation of discrete time signals

The concept of frequency is closely related to a specific type of periodic motion called harmonic oscillation, which is described by sinusoidal functions. The CT and DT sinusoidal signals are characterized by the following properties:

1. A continuous time sinusoid $x(t) = \cos(2\pi f_a t)$ is periodic for any value of f_a.

But for DT sinusoid $x[n]=\cos(2\pi f_d n)$ to be periodic with period N (an integer), we require

$$\cos(2\pi f_d n) = \cos[2\pi f_d(n+N)] = \cos(2\pi f_d n + 2\pi f_d N)$$

This is possible only if

$$2\pi f_d N = 2\pi k \;\; (k\ is\ an\ integer)$$

Or

$$f_d = \frac{k}{N}$$

i.e., the discrete frequency f_d must be a rational number (ratio of two integers).

Similarly, a discrete time exponential $e^{j\omega n}$ is periodic only if $\frac{\omega}{2\pi} = f_d = rational\ number.$

The period is the denominator after $\frac{\omega}{2\pi}$ is simplified such that in $\frac{\omega}{2\pi} = \frac{k}{N}$, k and N are relatively prime.

2. A CT sinusoidal signal x(t) = cos(Ωt) has a unique waveform for every value of Ω, 0 < Ω < ∞. Increasing Ω results in a sinusoidal signal of ever – increasing frequency.
   But, for a DT sinusoidal signal cos (ωn), considering two frequencies separated by an integer multiple of 2π, (ω and ω $\pm$2πm, m is an integer), we have
   $$\cos[(\omega \pm 2\pi m)n] = \cos(\omega n \pm 2\pi mn)$$
   Since m and n are both integers
   $$\cos(\omega n \pm 2\pi mn) = \cos(\omega n)$$
   So, a DT sinusoidal sequence has unique waveform only for the values of ω over a range of 2π. The range $-\pi \leq \omega \leq \pi$ defines the fundamental range of frequencies or principal range.
3. The highest rate of oscillation in a DT sinusoidal sequence is attained when ω=π or ω= - π . the rate of oscillation increases continually as ω increases from 0 to π, then decreases as ω increases from π to 2π. So low – frequency DT sine waves have ω near 0 or any even multiple of π, while the high – frequency sine waves have ω near $\pm$ π or other odd multiples of π.

Frequency domain representation of discrete time systems

The frequency response function completely characterizes a linear time invariant system in the frequency domain. Since, most signals can be expressed in Fourier domain as a weighted sum of harmonically related exponentials, the response of an LTI system to this class of signals can be easily determined.

The response of any relaxed LTI system to an arbitrary input signal x[n] is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Here, the system is characterized in the time domain by its impulse response h[n]. to develop a frequency domain characterization of the system, we excite the system with the complex exponential

$$x[n] = Ae^{j\omega n}, -\infty < n < \infty$$

Where A is the amplitude and ω is any arbitrary frequency confined to the frequency interval [- π, π]. By substituting this in the above convolution sum, we obtain the response as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\left[Ae^{j\omega(n-k)}\right]$$

$$= A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n}$$

Here, the term inside the brackets is a function of frequency ω. It is the Fourier Transform of the impulse response h[n], and is denoted by

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

And $y[n] = AH(\omega)\ e^{j\omega n}$

Since the output differs from the input only by a constant multiplicative factor, the exponential input signal is called the **eigen function** of the system, and the multiplicative factor is called the **eigenvalue** of the system.

H(ω) is a complex valued function of the frequency variable ω.